INVESTMENT UNDER UNCERTAINTY:

STATE PRICES IN INCOMPLETE MARKETS

A DISSERTATION

SUBMITTED TO THE DEPARTMENT OF

ENGINEERING-ECONOMIC SYSTEMS AND OPERATIONS RESEARCH

AND THE COMMITTEE ON GRADUATE STUDIES

OF STANFORD UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

Thomas Edward Hoff

August, 1997

I certify that I have read this dissertation and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

John P. Weyant (Principal Advisor)

I certify that I have read this dissertation and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

James L. Sweeney

I certify that I have read this dissertation and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

David G. Luenberger

Approved for the University Committee on Graduate Studies:

Abstract

An investment's price is the state-price weighted sum of its future payoffs when markets are complete. This is a well established fact in the financial economics field. The investment valuation problem in incomplete markets, however, has attracted the interest of several additional fields, including decision analysis and real options, and has led to a variety of valuation approaches. The approaches differ in their treatment of the investor's existing portfolio, market opportunities to hedge risks, the ability to re-optimize after adding a new investment to one's portfolio, and the investment's divisibility.

This research develops a single valuation approach that produces results consistent with financial economics when markets are complete but is also applicable when the investment is not divisible and the decision maker can only borrow and lend at the risk-free discount rate (markets are incomplete). Results suggest that, given a time- and state-separable utility function, a decision maker's buying price for an investment is approximately equal to the state-price weighted sum of its future payoffs (see Theorems 2.2 and 3.2); results are exact when markets are complete or the utility function is exponential. State prices in incomplete markets have a similar definition as state prices in complete markets in that they are approximately marginal utility based prices. The approach may have computational advantages because the prices can be estimated without solving a full utility maximization problem for each new investment that is evaluated.

The most important application of this work will be in the evaluation of projects that are indivisible and have managerial flexibility in markets that are incomplete (i.e., an important category of real option problems). A comprehensive example of an electrical utility's use of distributed generation to provide system capacity (rather than the typical approach of upgrading transmission and distribution facilities) illustrates how to apply the method. The example shows that the correlation between a new project's payoffs and the existing portfolio has a large effect on the investment decision.

Acknowledgements

Many people have helped me during my studies at Stanford. The most important is my advisor John Weyant. John has been a caring, committed, and knowledgeable advisor and a good friend throughout my time at Stanford.

My committee members, Jim Sweeney and David Luenberger, have also been helpful. Jim Sweeney set a good example of to wed academic theories with real life problems, encouraged me to produce my best research, and provided good feedback and direction throughout my research. I have benefited from David Luenberger's leadership in the investment science area as well as from his other books.

I was helped by discussions at various stages of my research with people from Stanford including Darrell Duffie, Ron Howard, Blake Johnson, Karl Knapp, Gil Masters, Ross Shachter, and Tom Varner, and those outside the university including Shimon Awerbuch (Independent Economist), Steve Chapel (Electric Power Research Institute), Chuck Feinstein (Santa Clara University), Spiros Martzoukas (George Washington University), Pete Morris (Applied Decision Analysis), Ren Orans (Energy and Environmental Economics), Jamie Reed (Incentives Research), Tom Rutherford (University of Colorado at Boulder), Jim Smith (Duke University), Elizabeth Teisberg (University of Virginia), and Howard Wenger (Pacific Energy Group). Staff members Maggie Barstow Taylor, Susan Clement, Nancy Florence, Danielle Herrmann, Pam McCrosky, and Roz Morf have been supportive throughout the process.

Above all, I am thankful to God for the opportunity to have come to Stanford with the support of friends and family. A special thanks to my wife, Elaine, my children, Rachel, Jonathan, David, and Samuel, and my parents, Sam and Jo Hoff.

Table of Contents

List of Figures

1. Introduction

1.1 Background

Identifying better ways to evaluate investments is an activity in which there is widespread interest. It is of interest to individuals with very limited resources as well as to governments with vast financial resources. Individuals face investment problems when purchasing a washing machine, when deciding whether to invest in a college education, when investing in the stock market, and when buying a home. Firms face investment problems when designing their research and development programs, when introducing new products, and when making bet-the-company type decisions. Governments face investment problems when designing welfare programs, when supporting the development of new technologies, and when deciding whether or not to enter into an international confrontation.

This interest in investment problems has attracted the attention of several fields of research and has led to a variety of valuation theories and approaches. Two of the most important fields are financial economics and decision analysis/decision sciences. A third that is growing in importance is the field of real options.

One issue that tends to distinguish these three fields is their focus. As illustrated in Figure 1.1, financial economics focuses on the market. It provides a market-based valuation of an investment by examining all investment opportunities that are available in the market. Decision analysis focuses on the decision maker. Little thought is typically given to the market and how it affects the decision. Real options focuses on the investment. Some applications of real options are concerned with how the investment interacts with the market while others are only concerned about the decision maker. In some ways, it can be viewed as a field that is in between financial economics and decision analysis.

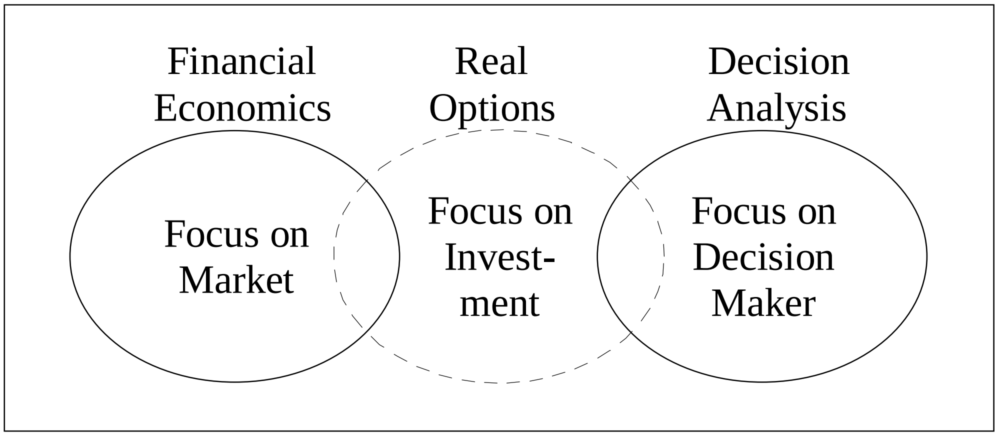

Figure 1-1. Focus of various fields when evaluating investments.

This chapter provides a brief discussion of these fields with a focus on some of the issues that will apply to the rest of this work.

1.2 Financial Economics

The existence of state prices is the unifying principle in asset pricing in financial markets. If state prices exist, there is one price for each state of the world at each date. State prices take into account uncertainty and discounting over time. A state price is the current price of a security that pays off \$1 if a particular state occurs in the future and pays off \$0 in all other states. The price of any security is determined according to these state prices and the security's cash flows. Security price is the state-price weighted sum of its future cash flows. That is, one multiplies the state price times the security's cash flow for each state and for each date and then sums the results.

Determination of state prices is based on the constraints placed on asset prices. The three basic constraints on asset prices are the absence of arbitrage (i.e., there is no way to lock in a risk-free profit by simultaneously entering into two or more market transactions), market equilibrium, and single-agent optimality. State prices exist when any of these three constraints is satisfied. That is, the absence of arbitrage, market equilibrium, and single-agent optimality each imply the existence of a set of state prices and complete markets (Duffie 1992). The following subsections describe how to determine state prices using these three constraints.

1.2.1 Valuation by Arbitrage

Valuation by arbitrage determines the prices of derivative securities based on the prices of other securities that are traded in the market. (A derivative security is a security whose value depends on the values of other more basic underlying variables). The arbitrage approach takes the price processes of a set of market-traded securities as given, demonstrates that they are free from arbitrage, and then uses this arbitrage-free set of prices to price the derivative security.

The best known option pricing formulas based on arbitrage are those developed by Black and Scholes (1973) and the binomial option pricing method developed by Cox, Ross, and Rubinstein (1979). The Black-Scholes approach prices a stock option by creating a dynamically hedged portfolio that consists of the underlying stock and a risk-free asset. The portfolio is selected so that its cash flow is identical to the option's cash flow in every state. Securities with identical cash flows have the same price to avoid arbitrage. Thus, the option's price is identical to the portfolio's price.

The binomial option pricing approach (1) generates the distribution of the future stock price based only on the risk-free rate and the stock's volatility (the risk premium of the stock is irrelevant); (2) calculates the expected value of the stock price minus the exercise price in the range where this value is positive using a set of "risk-neutral" probabilities; and (3) discounts the result at the risk-free rate. The state prices are identical to the "risk-neutral" probabilities times the risk-free discount rate.

Harrison and Kreps (1979) demonstrate that martingales provide a direct way to price derivative assets based on these prices. Y is a martingale if the expected value of Y at any future time given the current available information equals the current value of Y. More precisely, a process $Y = \{Y(t); t = 0,1,\ldots,T\}$ is a martingale adapted to an information structure $F = \{\mathbf{F}_t; t = 0,1,\ldots,T\}$ if $E\left[Y(s)|\mathbf{F}_t\right] = Y(t)$ for all $s \geq t$ where $E\left[\cdot|\mathbf{F}_t\right]$ is the expectation conditional on $\mathbf{F}_t$. The probabilities in the expectation that make Y a martingale are the state prices normalized by the risk-free discount rate.

The necessary and sufficient condition for price processes not to admit arbitrage opportunities is that they are related to martingales through a normalization and change of probability (Duffie 1992 and Huang and Litzenberger 1988). What is the normalization that makes a contingent claim (e.g., an option on a stock) into a martingale? Several authors have shown that any contingent claim on an asset can be priced in a world with systematic risk by replacing its actual growth rate with a certainty-equivalent rate (i.e., its risk-free rate) and then behaving as if the world were risk neutral (Cox and Ross 1976, Constantinides 1978, and Cox, Ingersoll, and Ross 1985). That is, the normalization in a market context is that a non-dividend paying stock earns at the risk-free rate (regardless of its risk), the expectation of the normalized stock price minus the exercise price is taken as if the agent is risk-neutral, and the result is discounted at the risk-free discount rate. In this case, the state prices are identical to the "risk-neutral" probabilities times the risk-free discount rate.

State prices in a two-period world can be determined using an arbitrage approach as follows. Suppose that the number of linearly independent securities (N) equals the number of states (S). The period 1 state-price vector is $\boldsymbol{\psi}_\mathbf{1} = \begin{bmatrix} \psi_1^1 & \psi_1^2 & \ldots & \psi_1^S \end{bmatrix}^T$; the period 0 security-price vector is $\boldsymbol{q} = \begin{bmatrix} q_1 & q_2 & \ldots & q_N \end{bmatrix}^T$; and the period 1 security cash flow matrix $\boldsymbol{D}$ is an N by S matrix where element D_{ij} is the cash flow of security i in state j. A system that is free from arbitrage requires that $\boldsymbol{q} = \boldsymbol{D}\boldsymbol{\psi}_\mathbf{1}$. $\boldsymbol{D}$ is invertible because the securities are linearly independent and markets are complete (i.e., $S = N$). Thus, the state price vector equals the inverse of the cash flow matrix times the period 0 security prices, or $\boldsymbol{\psi}_\mathbf{1} = \boldsymbol{D}^{-1}\boldsymbol{q}$.

1.2.2 Equilibrium Valuation

While valuation by arbitrage is the most widely used approach to pricing derivative securities in financial markets, it is not the only approach. Another approach is equilibrium valuation. This approach was pioneered by Arrow (1964) and Debreu (1959). Rubinstein (1976) demonstrated how equilibrium valuation is linked to valuation by arbitrage.

Equilibrium valuation in a pure trade economy with complete markets begins with a group of agents. Each agent has an initial endowment and a strictly increasing, strictly concave, differentiable utility function, and optimizes consumption using a trading strategy such that the market is in equilibrium. The associated equilibrium allocation is Pareto optimal. This means that there is no other feasible allocation that makes all agents at least as well off and at least one agent better off. Since there is an equilibrium that is Pareto optimal, there is no arbitrage, and therefore there is a set of state prices (Duffie 1992).

The equilibrium approach differs from the arbitrage approach in that no prices are given initially. Rather, all prices in the economy are determined through an economic optimization. The result of the optimization is a set of state prices based on marginal utilities.

Huang and Litzenberger (1988) show that when the allocation of state contingent claims is efficient and individuals have time-additive, state-independent, strictly increasing, strictly concave, and differentiable utility functions, state prices are determined as if there were a single individual in the economy endowed with the aggregate endowment. State prices are the set of prices that make the representative agent's initial endowment the optimal consumption choice.

This problem is solved by maximizing the representative agent's utility maximization problem subject to its endowment, $\left\{e_0, \boldsymbol{e_1}; e_0 \in R_+^1, \boldsymbol{e_1} \in R_+^S\right\}$. The first order necessary conditions of the maximization problem $\underset{z_0, \boldsymbol{z_1}}{Max} \left[u_0(z_0) + \boldsymbol{p} \cdot \boldsymbol{u_1}(\boldsymbol{z_1})\right]$ subject to the constraint that $\psi_0 z_0 + \boldsymbol{\psi_1} \cdot \boldsymbol{z_1} \leq \psi_0 e_0 + \boldsymbol{\psi_1} \cdot \boldsymbol{e_1}$ along with market clearing conditions require that $\dfrac{\psi_1^i}{\psi_0} = \dfrac{p^i u_1'(e_1^i)}{u_0'(e_0)}$ for $1 \leq i \leq S$ ($\boldsymbol{p}$ is the probability vector $\boldsymbol{p} = \begin{bmatrix} p^1 & p^2 & \cdots & p^S \end{bmatrix}$ where p^i is the probability that state i will occur). Summing the first order necessary conditions over all i and treating ψ_0 as the numeraire results in the ratio of the period 1 expected marginal utility divided by the period 0 marginal utility being equal to the sum of state prices. That is, $\boldsymbol{\psi_1} \cdot \boldsymbol{1} = \dfrac{\boldsymbol{p} \cdot \boldsymbol{u_1}'(\boldsymbol{e_1})}{u_0'(e_0)}$. The sum of state prices equals the price of a risk-free asset (i.e., that asset has a cash flow of \$1 in all states in period 1), so that $\boldsymbol{\psi_1} \cdot \boldsymbol{1} = \psi_1^0$, where ψ_1^0 is the risk-free discount rate. This and the previous equation are substituted into the first order necessary conditions to obtain the set of state prices, namely that $\psi_1^i = \psi_1^0 \dfrac{p^i u_1'(e_1^i)}{\boldsymbol{p} \cdot \boldsymbol{u_1}'(\boldsymbol{e_1})}$ for $1 \le i \le S$. The set of $\dfrac{p^i u_1'(e_1^i)}{\boldsymbol{p} \cdot \boldsymbol{u_1}'(\boldsymbol{e_1})}$ for $1 \le i \le S$ are interpreted as the "risk-neutral" probabilities because $\sum_{i=1}^{S} \dfrac{p^i u_1'(e_1^i)}{\boldsymbol{p} \cdot \boldsymbol{u_1}'(\boldsymbol{e_1})} = 1$. Thus, the state prices equal the "risk-neutral" probabilities times the risk-free discount rate.

1.2.3 Valuation by Single-Agent Optimality

The least common approach to determine state prices is through the use of single agent optimality. This approach determines state prices by examining individual consumer behavior. It turns out that this may be the most important approach from the perspective of this research.

Given that an agent with a strictly increasing utility function is consuming optimally and markets are complete, the ratio of two state prices equals the ratio of the respective marginal utilities.[1] The mechanics and results of this approach are very similar to the equilibrium valuation approach. The primary difference in the result is that the individual agent's utility function replaces the representative agent's utility function and individual agent consumption replaces market endowment in the state prices.

The consumer's utility maximization problem can be formulated by allowing the consumer to purchase some portfolio of market-traded securities that have state-contingent payoffs that result in the optimal consumption over time. Alternatively, the problem can be formulated by taking the set of state prices as given[2] and then allowing the consumer to optimize state-by-state consumption; that is the approach taken here.

The first order conditions for optimality for the problem $\underset{c_0, \boldsymbol{c_1}}{Max}\ U(c_0, \boldsymbol{c_1})$ subject to the constraint that $c_0 + \boldsymbol{\psi_1} \cdot \boldsymbol{c_1} \le e_0 + \boldsymbol{\psi_1} \cdot \boldsymbol{e_1}$ are $\dfrac{\partial U}{\partial (c_0)} = \lambda$ and $\dfrac{\partial U}{\partial (c_1^i)} = \psi_1^i \lambda$ for $1 \le i \le S$.

---

[1] Luenberger (1997) shows that a similar result can be obtained when markets are incomplete as long as the investments are infinitely divisible.

[2] The prices could be calculated using valuation by arbitrage.

Summing over all *i*, $\frac{\partial \boldsymbol{U}}{\partial(\boldsymbol{c_1})} \cdot \mathbf{1} = \boldsymbol{\psi_1} \lambda \cdot \mathbf{1} = \psi_1^0 \lambda$. The binding budget constraint leads to the result that $\psi_1^i = \psi_1^0 \frac{\frac{\partial U}{\partial\left(c_1^i\right)}}{\frac{\partial \boldsymbol{U}}{\partial(\boldsymbol{c_1})} \cdot \mathbf{1}}$. If the utility function is an additive expected utility as in the previous section (i.e., $U(c_0, \boldsymbol{c_1}) = u_0(c_0) + \boldsymbol{p} \cdot \boldsymbol{u_1}(\boldsymbol{c_1})$) then the state prices have the same form as before (i.e., $\psi_1^i = \psi_1^0 \frac{p^i u_1'\left(c_1^i\right)}{\boldsymbol{p} \cdot \boldsymbol{u_1}'(\boldsymbol{c_1})}$). Once again, the state prices can be interpreted as the "risk-neutral" probabilities times the risk-free discount rate.

1.3 Real Options

Financial option evaluation methods have more recently been applied to evaluate the flexibility associated with physical investments. Some have labeled this extension real options. Real option evaluations account for the value of flexibility embedded within projects. Like the field of financial option valuation, this is a large and growing field.[3]

The field of real options is an important one from the perspective of this research because it represents a field that is between financial economics and decision analysis. The field of real options focuses on the investment and how to capture the value of the flexibility of the investment.

This focus is best illustrated by Dixit and Pindyck (1994). The authors take two approaches to evaluating investments. One is a contingent claims analysis and the other is a dynamic programming analysis. The contingent claims analysis, which is essentially the same as a financial economics options valuation, constructs a risk neutral portfolio and applies the principle of no arbitrage to value the investment. The critical assumption implicit in the contingent claims analysis is that stochastic changes in the investment's value are spanned by existing assets in the economy. This requires that capital markets are sufficiently complete so that a dynamic portfolio of assets could be constructed whose price is perfectly correlated with the value of the investment. This is a crucial point, because this assumption is widely made when evaluating real options. For example, this is a fundamental assumption in the book *Real Options* by Trigeorgis (1996).

Dixit and Pindyck value an investment using a dynamic programming approach when spanning conditions do not exist. The application of this approach states that, over a short interval of time, the total expected return of the investment opportunity is equal to its expected rate of capital appreciation. According to Dixit and Pindyck (1994, p. 147), a difficulty of this approach is that "it is based on an arbitrary and constant discount rate. It is not clear where this discount rate should come from, or even that it should be constant over time." Dixit and Pindyck (1994, p.152) elaborate on this critical point (with the emphasis being mine):

> "Hence, the contingent claims solution to our investment problem is equivalent to a dynamic programming solution, under the assumption of risk neutrality (that is the discount rate [of the dynamic programming approach] is equal to the risk-free rate). Thus, whether or not spanning holds, we can obtain a solution to the investment problem, but without spanning, the solution will be subject to an assumed discount rate. In either case, the solution will have the same form, and the effects of changes [in certain key variables] will likewise be the same. One point is worth noting, however. ***Without spanning, there is no theory for determining the 'correct' value for the discount rate*** (unless we make restrictive assumptions about investors' or managers' utility functions). ***The CAPM, for example, would not hold, and so it could not be used to calculate a risk-adjusted discount rate in the usual way."***

The implication of this is that, while you can use the dynamic programming approach when markets are incomplete, there is no theoretically correct way to select the correct discount rate.

1.4 Decision Analysis

The field of decision analysis (and decision sciences) focuses on the decision maker. A wide range of theories are captured under this title. Some of these theories include expected utility, subjective expected utility, prospect theory, rank-dependent utility, state-dependent subjective expected utility, etc. It would be difficult to give an adequate treatment to all of these theories in this section.

---

[3] See the bibliography in a recent book on the subject (Dixit and Pindyck, 1994) or Trigeorgis (1996).

The major point of this section is that all of these theories are united by their focus on the decision maker. Whether the theory is normative in that it is intended to suggest how decisions should be made or whether it is descriptive in that it is intended to describe how decisions are actually made, the central focus of all of these theories is on the decision maker.

Two theories are briefly discussed here: expected utility (and subjective expected utility) and prospect theory. The expected utility theories have been around the longest (Bernoulli 1738, and von Neumann and Morgenstern 1944) and are probably still the most widely used. Fishburn (1982) presents an excellent summary of the various axiomatic approaches to arrive at the expected utility theories. Howard (1992, pp. 33-34) takes another approach in his articulation of the foundations of decision analysis in five rules of thought. While there are a variety of ways the parameters in the expected utility function can be expressed, a typical one is that the utility of a project with payoffs of $\boldsymbol{x}$ equals $\sum_{i=1}^{S} p^i u\left(w_0 + x^i\right)$, where p^i is the probability of the payoff x^i occurring and w_0 is initial wealth. One wants to determine the certain equivalent CE that makes the decision maker's utility with the payoffs the same as the utility with the certain equivalent (i.e., $u\left(w_0 + CE\right) = \sum_{i=1}^{S} p^i u\left(w_0 + x^i\right)$).

A second theory that is of particular interest to this research is prospect theory (Kahneman and Tversky 1979). Prospect theory was developed in a response to observed descriptive violations of expected utility theory. Prospect theory suggested that the payoffs $\boldsymbol{x}$ be evaluated using a function of the form $\sum_{i=1}^{S} \varphi\left(p^i\right) u\left(x^i\right)$, where $\varphi\left(p^i\right)$ is a decision weight; future work extended the definition of the decision weight to depend on other variables as well as whether there was a loss or a gain (Hogarth and Einhorn 1990, Tversky and Wakker 1995). While proponents of prospect theory state that the decision weights are not probabilities, the improved descriptive capability of the inclusion of decision weights will be seen to be of value later in this research.

1.5 Method Comparison

To set the stage for the contribution that this research makes, it is important to identify some of the weaknesses associated with decision analysis as it is typically applied. As stated above, it would be difficult to do this for all of these theories. Fortunately, the three primary weaknesses identified in the following sections are applicable to essentially all of the theories. For this reason, the basis of comparison is expected utility theory.

1.5.1 Expected Utility Results can be Inconsistent with a Market Valuation

First, the valuation results from an expected utility approach are not necessarily consistent with a financial economics approach. Consider the following illustration. Suppose that a risk-averse decision maker is evaluating three projects. Project X pays x, which is greater than 0, with probability p else it pays 0 $(x, p; 0, 1-p)$. Project Y pays y, which is greater than 0, with probability 1-p else it pays 0 $(0, p; y, 1-p)$. Project X+Y is the sum of project X and project Y so that it pays x with probability p and y with probability 1-p $(x, p; y, 1-p)$. Let V_X be the value for the financial economics approach and the certain equivalent for the expected utility approach of project X. The value of two projects combined equals the values of the individual projects summed together when using a financial economics approach, i.e., $V_{X+Y} = V_X + V_Y$. The certain equivalent of two projects combined, however, does not necessarily equal the certain equivalents of the individual projects summed together when using an expected utility approach.

The financial economics approach starts with the assumption of complete markets. This means that a state-price vector exists. Thus, the state-price weighted sum off the payoffs for project X is $V_X = \begin{bmatrix} x & 0 \end{bmatrix} \cdot \begin{bmatrix} \psi^1 & \psi^2 \end{bmatrix} = \psi^1 x$. Likewise, $V_Y = \begin{bmatrix} 0 & y \end{bmatrix} \cdot \begin{bmatrix} \psi^1 & \psi^2 \end{bmatrix} = \psi^2 y$ and $V_{X+Y} = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} \psi^1 & \psi^2 \end{bmatrix} = \psi^1 x + \psi^2 y$. This means that $V_{X+Y} = V_X + V_Y$.

The expected utility approach requires that $u(w_0 + V_X) = pu(w_0 + x) + (1-p)u(w_0)$ and $u(w_0 + V_Y) = pu(w_0) + (1-p)u(w_0 + y)$ for projects X and Y. This means that

$V_X = u^{-1}\left[pu(w_0 + x) + (1-p)u(w_0)\right] - w_0$ and $V_Y = u^{-1}\left[pu(w_0) + (1-p)u(w_0 + y)\right] - w_0$.[4] Suppose that $V_{X+Y} = V_X + V_Y$. Substituting for $V_X + V_Y$, adding w_0 to both sides, and taking the utility results in $u(w_0 + V_{X+Y}) = u\left\{u^{-1}\left[pu(w_0 + x) + (1-p)u(w_0)\right] + u^{-1}\left[pu(w_0) + (1-p)u(w_0 + y)\right] - w_0\right\}$. The expected utility approach requires that $u(w_0 + V_{X+Y}) = pu(w_0 + x) + (1-p)u(w_0 + y)$ for the project *X+Y*. The previous equation simplifies to this only when $u(A+B) = u(A) + u(B)$. This is the case when u is linear and the decision maker is risk-neutral. This violates the initial assumption that the decision maker is risk-averse. Thus, an expected utility approach for a risk-averse decision maker cannot obtain a value for all three of these projects that is consistent with a financial economics approach.

1.5.2 Expected Utility Theory Lacks a Temporal Element

Second, an expected utility formulation lacks a temporal element although the parameters used in the analysis are often from different time periods. Specifically, the expected utility analysis evaluates the expected utility of wealth (at the time when the decision is made) plus the project's payoff (at the time when the uncertainty is resolved). A difficulty with this formulation is that the existence of uncertainty in any decision problem requires that there is some period of time between when the decision is made and the uncertainty is resolved (Pope 1985). For example, if the decision is made in period 0 and the uncertainty is resolved in period 1, wealth is taken at period 0 but the project's payoff is taken at period 1.

One solution to this problem is to take wealth and payoff from the same period with the appropriate period being the time when the payoff occurs. That is, wealth is taken from period 1 rather than period 0. An implication of this is that wealth in the expected utility formulation is no longer certain but can vary. The payoff itself is no longer the key factor. Rather, it is the covariance between wealth and payoff that is crucial.

---

[4] It is assumed that the decision maker's utility function is invertible.

Kasanen and Trigeorgis (1995) formulate a decision analytic problem using this approach of having wealth and the payoffs occur at the same time. They assume that there exists a utility function for the whole economy. They then take a first order Taylor series expansion of their formulation around market wealth. The result they derive is the same as one would obtain by taking the state prices defined using an equilibrium approach (section 1.2.3), replacing endowment with wealth, calculating the state-price weighted sum of the payoffs, and applying the definition of covariance. That is, they show that an expected utility framework can be linked to finance theory by allowing wealth to vary.

1.5.3 Expected Utility Theory Does Not Allow for Prior Consumption Changes

Third, expected utility theory does not allow for changes in prior consumption that can occur when a certain equivalent is given rather than a project. That is, offering a certain equivalent for the project's payoffs in isolation from other decisions can alter decisions that must be made before the uncertainty is resolved. Matheson and Howard (1989, p. 44) recognize this when they state that "the approach [of calculating the certain equivalent of the project] is appropriate when there is no opportunity to utilize the information about the outcomes as it is revealed." Likewise, Keeney and Raiffa (1976, p. 512) point out that the time resolution of uncertainty affects earlier acts, and Becker and Sarin (1989) and LaValle (1989, 1992) state that decision trees cannot be simplified by certain equivalent substitutions without potentially affecting preferences for initial acts. Viewed from an economics perspective, Mossin (1969), Spence and Zeckhauser (1972), and Dreze and Modigliani (1972) observe that induced preference for income will not in general satisfy the von Neumann-Morgenstern axioms even if preference for consumption has an expected utility representation.

In response to some of these observations, Kreps and Porteus (1978) propose a generalization of von Neumann-Morgenstern utility called temporal von Neumann-Morgenstern preference. Kreps and Porteus (1979a) derive the necessary and sufficient conditions for induced preference to satisfy the von Neumann-Morgenstern axioms in a two-period model and show that they are quite stringent. In a consumption-savings problem, the conditions translate into a utility function of the form

$U(c_0, c_1) = f(c_0) + g(c_0)h(c_0 + c_1)$, a special case of which is a utility function that is exponential in period 1: $U(c_0, c_1) = u_0(c_0) - \exp(-\lambda c_1)$. Smith and Nau (1995) extend this result to partially complete markets with more than two time periods with similar results (i.e., there is a time-separable utility function that is exponential in every time period except period 0).

1.6 Objective

The objective of this research is to develop a valuation approach that is theoretically consistent whether markets are complete or incomplete and investments are not infinitely divisible. The only things that will change depending upon market conditions are the parameter inputs into the valuation framework. That is, the goal is to provide an evaluation framework that produces results that are identical to a financial economics approach when markets are complete but is applicable when markets are incomplete.

The outline of the report is as follows. Chapter 2 develops the approach in a discrete time, two-period world. Chapter 3 extends the results to multiple time periods. Chapter 4 illustrates how to apply the method to a real world problem. Conclusions and recommendations for further research are presented in Chapter 5. Proofs are presented in Chapter 6 and a discussion about time- and state-separable utility functions is given in Chapter 7.

2. Single-Period Model

This chapter develops a single valuation approach applicable in complete and incomplete markets in a discrete time, single-period setting. Assumptions and definitions are presented in Section 2.1. A decision maker's buying price for an investment with uncertain payoffs satisfies the condition that utility with the net payoffs (see Definition 2.2) equals utility without the net payoffs (Section 2.2) and the condition that utility cannot be improved by changing consumption or market transactions (Section 2.3). An example is included in Section 2.2 about how this approach eliminates the need to separately define a "buying price" and a "selling price." Section 2.4 introduces the assumption of a time- and state-separable utility function and demonstrates that the decision maker's buying price for the investment is approximately equal to the state-price weighted sum of its payoffs; results are exact when markets are complete or the utility function is additive exponential. Exponential and logarithmic examples are included.

2.1 Setting

This section assumes a discrete time, single-period setting. Period 0 has no uncertainty and period 1 has a finite set, $\{1, \quad 2, \quad \ldots, \quad S\}$, states of uncertainty, one of which will be revealed to be true in period 1. Complete markets means that a unique state-price vector exists with a price for all states (Duffie 1992). Incomplete markets means that the only asset available is one that allows risk-free borrowing and lending. Both types of markets allow for the purchase of infinitely divisible assets. All vectors in this section have S elements unless specifically noted and are printed in **bold type**. $\boldsymbol{z} \in R^S$ means that $\boldsymbol{z}$ has S elements so that $\boldsymbol{z} = \begin{bmatrix} z^1 & z^2 & \ldots & z^S \end{bmatrix}$ and R^S means that there are no sign restrictions on the elements of $\boldsymbol{z}$; $\boldsymbol{z} \in R_+^S$ means that every element of $\boldsymbol{z}$ is non-negative; and $\boldsymbol{z} \in R_{++}^S$ means that every element of $\boldsymbol{z}$ is strictly positive. Subscripts refer to times and superscripts refer to states.

Definition 2.1: $S_0 \in R^1$ is the period 0 price of an investment that has period 1 payoffs of $\boldsymbol{x_1} \in R^S$. Uncertainty is resolved in period 1 and there are no sign restrictions on the investment price or its payoffs. S_0 may be known or unknown. The objective of this work is to either find S_0 if it is unknown or to determine if the investment should be made if it is known.

Definition 2.2: $B_1 \in R^1$ is the decision maker's period 1 buying price for the investment's period 1 payoffs $\boldsymbol{x_1}$. It is the amount that the decision maker is willing to pay in period 1 in order to receive the payoff vector $\boldsymbol{x_1}$. This definition is made more precise in Theorem 2.1. The net payoffs are the payoffs minus the period 1 buying price; i.e., $\boldsymbol{x_1} - B_1\mathbf{1} \in R^S$. The period 0 buying price equals the period 1 buying price discounted at the risk-free rate since the period 1 buying price is a certain cash flow; i.e., $B_0 = \psi_1^0 B_1$, where ψ_1^0 is the risk-free discount factor (0 superscript) between period 0 and period 1 (1 subscript). The decision maker is better off buying the investment if the price is less than the buying price; i.e., $S_0 < B_0$. The price does not have to equal the buying price in incomplete markets.

Definition 2.3: $\Delta c_0 \in R^1$ and $\Delta \boldsymbol{w_1} \in R^S$ are the changes in period 0 consumption and period 1 wealth due to re-optimization after the net payoffs are added. For example, suppose that an initial optimization states that optimal wealth in period 1 state i is w_1^i. The addition of the investment's payoff in state i as well as the buying price for the investment changes this to $w_1^i + x_1^i - B_1$. The decision maker then re-optimizes and the new wealth is $w_1^i + x_1^i - B_1 + \Delta w_1^i$.

Assumption 2.1: The decision maker has a strictly increasing utility function that maps period 0 consumption and period 1, state-dependent wealth to a real number. $U: R_+ \times R_+^S \to R$ where $U = U(z_0, \boldsymbol{z_1})$. $z_0 \in R_+^1$ is period 0 consumption and $\boldsymbol{z_1} \in R_+^S$ is period 1, state-dependent wealth. A solution exists to the utility maximization problem, $\max\limits_{z_0, \boldsymbol{z_1}} U(z_0, \boldsymbol{z_1})$, where $(z_0, \boldsymbol{z_1})$ is budget-feasible. A solution also exists when any finite uncertain payoff is offered to the decision maker.

Assumption 2.2: As illustrated in Panel 0 of Figure 2-1, the decision maker begins with an initial wealth w_0 and possibly a set of other pre-existing uncertain payoffs summarized by $\boldsymbol{X_1} \in R^S$; note that $\boldsymbol{X_1}$ is not the same as $\boldsymbol{x_1}$. The decision maker maximizes utility given these initial conditions prior to adding the net payoffs. The result is that the optimal period 0 consumption/period 1 wealth pair is $(c_0, \boldsymbol{w_1})$ before the net payoffs are added. The decision maker then adds in the net payoffs and re-optimizes with the result that the optimal period 0 consumption/period 1 wealth pair is $(\hat{c}_0, \boldsymbol{\hat{w}_1})$, where $\hat{c}_0 = c_0 + \Delta c_0$ and $\boldsymbol{\hat{w}_1} = \boldsymbol{w_1} + \Delta \boldsymbol{w_1} + \boldsymbol{x_1} - B_1\mathbf{1}$.

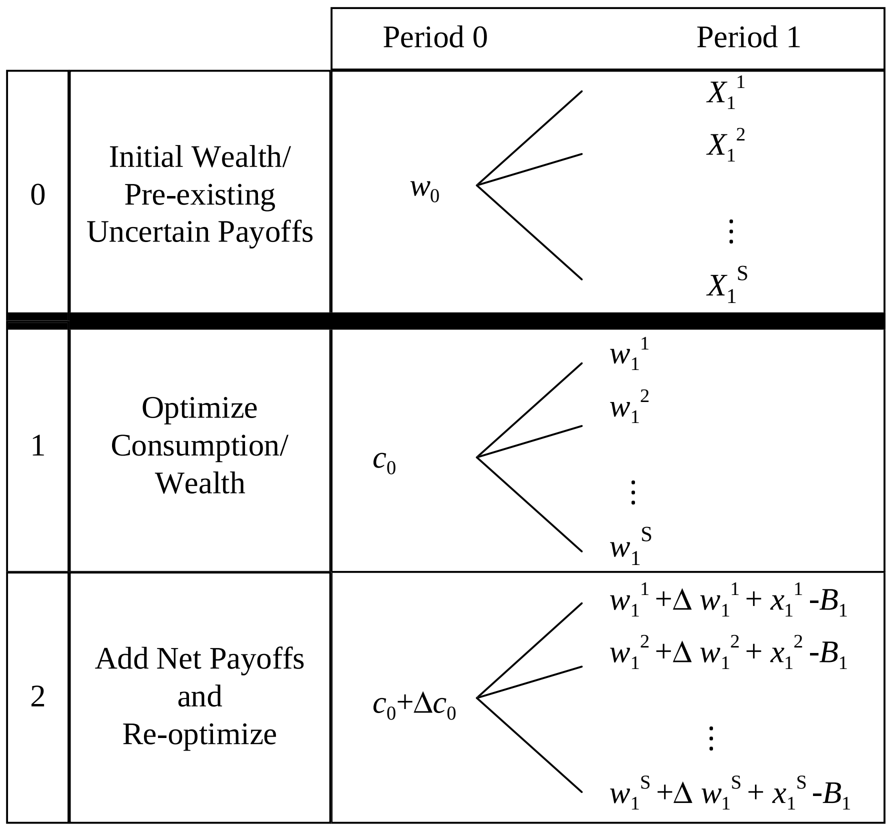

Figure 2-1. Initial condition and optimal cons./wealth with and without net payoffs.

2.2 Indifference Condition

Two conditions must be satisfied in order for B_1 to be the buying price of $\boldsymbol{x_1}$. First, the decision maker's utility with the net payoffs must be the same as the decision maker's utility without the net payoffs. Second, the decision maker cannot improve utility by changing consumption or market transactions. The first condition is developed in this section and the implications of the second condition are developed in Section 2.3.

Theorem 2.1: If B_1 is the period 1 buying price for the payoffs $\boldsymbol{x_1}$ then the utility of the original optimal consumption/wealth pair plus the net payoffs plus changes in consumption/wealth due to re-optimization (Panel 2 of Figure 2-1) equals the utility of the original consumption/wealth pair (Panel 1 of Figure 2-1).

$$U(\hat{c}_0, \hat{\boldsymbol{w}}_1) = U(c_0, \boldsymbol{w}_1) \tag{2.1}$$

where $\hat{c}_0 = c_0 + \Delta c_0$ and $\hat{\boldsymbol{w}}_1 = \boldsymbol{w}_1 + \Delta \boldsymbol{w}_1 + \boldsymbol{x}_1 - B_1 \mathbf{1}$.

Theorem 2.1 differs from a typical expected utility approach in four ways. First, there are no constraints on the utility function's form or the separability of the arguments. Second, wealth occurs in period 1 and can be state-dependent. Third, the decision maker can re-optimize after the net payoffs are added. Fourth, all changes to the utility function's arguments occur on one side of the equation by adding the payoffs, subtracting the buying price, and allowing the decision maker to re-optimize. This fourth point is attractive because there is no need to formulate separate problems depending upon whether the decision maker is "buying" or "selling" the investment; all problems are formulated as if the investment has been purchased. Consider the following example.

Example 2.1: "Buy Investment." A decision maker is deciding whether or not to pay \$26K in period 0 for an investment that has period 1 payoffs of \$20K or \$40K. Without the investment, optimal period 0 consumption is \$10K and period 1 wealth will be \$100K in all states. Thus, $\boldsymbol{w}_1 = [100 \quad 100]$ and $\boldsymbol{x}_1 = [20 \quad 40]$. As shown in the left side of Figure 2-2, Theorem 2.1 requires that $U(10 + \Delta c_0, 120 + \Delta w_1^1 - B_1, 140 + \Delta w_1^2 - B_1)$ = U(10,100,100). The decision maker will be better off to "buy" the investment if $\$26\text{K} < \psi_1^0 B_1$, where ψ_1^0 is the risk-free discount factor between period 0 and period 1.

"Sell Investment." Conversely, assume a decision maker owns an investment that will pay off either \$20K or \$40K in period 1 in addition to other period 0 wealth. The decision maker has been offered \$26K in period 0 for the investment and is deciding whether or not to accept the offer. After optimizing (but before calculating the buying price for the payoffs), the decision maker decides that optimal period 0 consumption is \$10K and period 1 wealth will be \$120K or \$140K. Thus, $\boldsymbol{w}_1 = [120 \quad 140]$ and $\boldsymbol{x}_1 = [-20 \quad -40]$. As shown in the right side of Figure 2-2, Theorem 2.1 requires that

$U\left(10+\Delta c_0, 100+\Delta w_1^1 - B_1, 100+\Delta w_1^2 - B_1\right) = U(10,120,140)$. The decision maker will be better off to "sell" the investment if $-\$26\text{K} < \psi_1^0 B_1$ (i.e., if $\$26\text{K} > -\psi_1^0 B_1$).

	"Buy Investment"		"Sell Investment" (i.e., buy negative investment)	
	Period 0	Period 1	Period 0	Period 1
Original $c_0/\boldsymbol{w_1}$	10	100 100	10	120 140
Original $c_0/\boldsymbol{w_1}$ + Net Payoffs + Re-optimization	10 $+\Delta c_0$	$120+\Delta w_1^{\,1}$ $-B_1$ $140+\Delta w_1^{\,2}$ $-B_1$	10 $+\Delta c_0$	$100+\Delta w_1^{\,1}$ $-B_1$ $100+\Delta w_1^{\,2}$ $-B_1$

Figure 2-2. Optimal consumption/wealth with and without net payoffs (Example 2.1).

2.3 Optimality Condition

The second condition that must be satisfied when there is an inter-temporal component to the utility function (i.e., there exists utility associated with period 0 consumption) or when markets are complete is that the decision maker cannot improve utility by changing consumption or market transactions. This condition is true by assumption and has several implications as summarized in the following corollaries; proofs for the corollaries are in the first Appendix.

Corollary 2.1: The optimality condition implies that period 0 marginal utility discounted at the risk-free rate between period 0 and period 1 minus the sum of period 1 marginal utilities equals zero.

$$\psi_1^0 \frac{\partial U}{\partial \hat{c}_0} - \frac{\partial \boldsymbol{U}}{\partial \hat{\boldsymbol{w}}_1} \cdot \mathbf{1} = \mathbf{0} \tag{2.2}$$

where $\frac{\partial \boldsymbol{U}}{\partial \hat{\boldsymbol{w}}_1} = \begin{bmatrix} \frac{\partial U}{\partial \hat{w}_1^1} & \frac{\partial U}{\partial \hat{w}_1^2} & \cdots & \frac{\partial U}{\partial \hat{w}_1^S} \end{bmatrix}$ and ψ_1^0 is the risk-free discount factor between period 0 and period 1.

Corollary 2.2: When markets are incomplete, the change in period 1 wealth due to re-optimization is constant across all states and equals the opposite of the escalated change in period 0 consumption .

$$\Delta w_1 = -\Delta c_0 / \psi_1^0 \ . \tag{2.3}$$

Corollary 2.3: When markets are complete, there is no change in period 0 consumption (so that $\hat{c}_0 = c_0$) or total period 1 wealth with the net payoff (so that $\hat{w}_1^i = w_1^i$ for $1 \le i \le S$) .

2.4 Buying Price

Assumption 2.3: The utility function is assumed to be time- and state-separable for the remainder of this section.

$$U(z_0, \boldsymbol{z_1}) = u_0(z_0) + \boldsymbol{u_1}(\boldsymbol{z_1}) \cdot \mathbf{1} \tag{2.4}$$

where $\boldsymbol{u_1}(\boldsymbol{z_1}) = \left[u_1^1(z_1^1) \quad u_1^2(z_1^2) \quad \ldots \quad u_1^S(z_1^S) \right]$.

This assumption is more general than assuming that the form of the utility function is expected utility (i.e., $\boldsymbol{u_1}(\boldsymbol{z_1}) = \left[p^1 u_1(z_1^1) \quad p^2 u_1(z_1^2) \quad \cdots \quad p^S u_1(z_1^S) \right]$) because it allows the form of the utility function to be state-dependent.

Theorem 2.2: For time- and state-separable utility functions, the period 0 buying price B_0 for period 1 payoffs of $\boldsymbol{x_1}$ is approximately equal to the state-price weighted sum of the payoffs. Results are exact when markets are complete or the period 1 utility function is linear or exponential.

$$B_0 \cong \boldsymbol{\psi_1} \cdot \boldsymbol{x_1} \ . \tag{2.5}$$

The state prices equal $\boldsymbol{\psi_1} = \dfrac{\psi_1^0 \nabla \boldsymbol{u_1}}{\nabla \boldsymbol{u_1} \cdot \mathbf{1}} \in R_{++}^S$, where $\nabla \boldsymbol{u_1} = \left[\nabla u_1^1 \quad \nabla u_1^2 \quad \ldots \quad \nabla u_1^S \right]$ and

$$\nabla u_1^i \left\{ \begin{array}{ll} = \dfrac{u_1^i(\hat{w}_1^i) - u_1^i(w_1^i)}{\Delta w_1^i + x_1^i - B_1} & \text{for } \Delta w_1^i + x_1^i - B_1 \neq 0 \\ = u_1^{i\,\prime}(w_1^i) & \text{for } \Delta w_1^i + x_1^i - B_1 = 0 \end{array} \right\}.$$

Proof: According to Theorem 2.1 for a time- and state-separable utility function, B_1 is the period 1 buying price for payoffs x_1 if $u_0(\hat{c}_0) + \boldsymbol{u_1}(\hat{\boldsymbol{w}}_1) \cdot \mathbf{1} = u_0(c_0) + \boldsymbol{u_1}(\boldsymbol{w_1}) \cdot \mathbf{1}$. This can be rewritten as $u_0(\hat{c}_0) - u_0(c_0) + [\boldsymbol{u_1}(\hat{\boldsymbol{w}}_1) - \boldsymbol{u_1}(\boldsymbol{w_1})] \cdot \mathbf{1} = 0$ and then as $\nabla u_0 \Delta c_0 + \nabla \boldsymbol{u_1} \cdot (\Delta \boldsymbol{w_1} + \boldsymbol{x_1} - B_1 \mathbf{1}) = 0$ with $\nabla \boldsymbol{u_1}$ defined above and $\nabla u_0 \begin{Bmatrix} = \dfrac{u_0(\hat{c}_0) - u_0(c_0)}{\Delta c_0} & \text{for } \Delta c_0 \neq 0 \\ = u_0{}'(c_0) & \text{for } \Delta c_0 = 0 \end{Bmatrix}$. Dividing by $\nabla \boldsymbol{u_1} \cdot \mathbf{1}$ (it is strictly positive because every element of $\nabla \boldsymbol{u_1}$ is strictly positive) and adding B_1 results in $B_1 = \left[\dfrac{\nabla \boldsymbol{u_1}}{\nabla \boldsymbol{u_1} \cdot \mathbf{1}}\right] \cdot \boldsymbol{x_1} + \left[\dfrac{\nabla \boldsymbol{u_1}}{\nabla \boldsymbol{u_1} \cdot \mathbf{1}}\right] \cdot \Delta \boldsymbol{w_1} + \dfrac{\nabla u_0}{\nabla \boldsymbol{u_1} \cdot \mathbf{1}} \Delta c_0$. B_1 is discounted to period 0 at the risk-free rate since it occurs at period 1 with certainty with the result that the period 0 buying price is

$$B_0 = \boldsymbol{\psi_1} \cdot \boldsymbol{x_1} + \left[\frac{\psi_1^0 \nabla \boldsymbol{u_1}}{\nabla \boldsymbol{u_1} \cdot \mathbf{1}}\right] \cdot \Delta \boldsymbol{w_1} + \frac{\psi_1^0 \nabla u_0}{\nabla \boldsymbol{u_1} \cdot \mathbf{1}} \Delta c_0 \quad . \tag{2.6}$$

Theorem 2.2 is proven if the second and third terms of (2.6) are approximately equal to zero; i.e., when $\left[\dfrac{\psi_1^0 \nabla \boldsymbol{u_1}}{\nabla \boldsymbol{u_1} \cdot \mathbf{1}}\right] \cdot \Delta \boldsymbol{w_1} + \dfrac{\psi_1^0 \nabla u_0}{\nabla \boldsymbol{u_1} \cdot \mathbf{1}} \Delta c_0 \cong 0$. Consider two cases.

Case 1: Complete Markets. Corollary 2.3 implies that $\Delta c_0 = 0$ and states that $\hat{w}_1^i = w_1^i$ for $1 \leq i \leq S$ when markets are complete. The third term of (2.6) equals 0 because $\Delta c_0 = 0$. The second term equals zero because $\left[\dfrac{\psi_1^0 \nabla \boldsymbol{u_1}}{\nabla \boldsymbol{u_1} \cdot \mathbf{1}}\right] = \left[\dfrac{\psi_1^0 \boldsymbol{u_1}'(\boldsymbol{w_1})}{\boldsymbol{u_1}'(\boldsymbol{w_1}) \cdot \mathbf{1}}\right]$, which are the state prices from a financial economics approach and the state price weighted sum of the change in period 1 wealth must equal zero to avoid arbitrage opportunities.

Case 2: Incomplete Markets. The change in period 1 wealth associated with the re-optimization is the same across all states when markets are incomplete and, according to Corollary 2.2, it equals $\Delta w_1 = -\dfrac{\Delta c_0}{\psi_1^0}$. Substituting this into the second and third terms of (2.6) results in $\left[\dfrac{\psi_1^0 \nabla u_0 - \nabla \boldsymbol{u_1} \cdot \mathbf{1}}{\nabla \boldsymbol{u_1} \cdot \mathbf{1}}\right] \Delta c_0$. This term is approximately equal to zero since the numerator in the square brackets is approximately equal to the optimality condition from Corollary 2.1.

Linear and exponential utility functions represent a special case when markets are incomplete so that Equation (2.6) is satisfied exactly. This is because there is no change in period 0 consumption.

Corollary 2.4: When markets are incomplete, the change in period 0 consumption must be zero if and only if the period 1 utility function is linear or exponential (see Appendix for proof).

The following two examples illustrate how Theorem 2.2 can be applied.

Example 2.2: Investment Opportunity Using an Exponential Utility Function. A firm is deciding whether or not to invest \$0.6M in period 0 in a project that is equally likely to pay off either nothing or \$2.0M in period 1. The risk-free discount rate is 10 percent, the firm's utility function is $U(z_0, \boldsymbol{z}_1) = u_0(z_0) - p^1 e^{-z_1^1/1} - p^2 e^{-z_1^2/1}$, where z is in millions of dollars, and markets are incomplete.

Consider the cases when there is positive and negative correlation between payoffs and existing wealth (i.e., $\boldsymbol{x}_1 = [0.0 \quad 2.0]$ and $\boldsymbol{w}_1 = [10.0 \quad 12.0]$ or $\boldsymbol{w}_1 = [12.0 \quad 10.0]$) as well as the case when wealth is certain ($\boldsymbol{w}_1 = [11.0 \quad 11.0]$). State prices are calculated using Theorem 2.2 with the results that: the buying price when there is positive correlation is $\$0.1\text{M} = [0.86 \quad 0.05] \cdot [\$0.0\text{M} \quad \$2.0\text{M}]$; the buying price when wealth is certain is $\$0.5\text{M} = [0.65 \quad 0.26] \cdot [\$0.0\text{M} \quad \$2.0\text{M}]$; and the buying price when there is negative correlation is $\$1.3\text{M} = [0.26 \quad 0.65] \cdot [\$0.0\text{M} \quad \$2.0\text{M}]$.[5] This suggests that the firm should only invest when there is negative correlation because the project acts as a hedge against uncertainty (i.e., this is the only case when the investment price is less than the buying

[5] The definition of state prices in (2.5) for a period 1 exponential utility function results in a state-price vector of $\psi_1 = \psi_1^0 \left[\frac{x_1^2 - B_1}{x_1^2 - x_1^1} \quad \frac{x_1^1 - B_1}{x_1^1 - x_1^2} \right]$, where $B_1 = -\rho \ln\left(\frac{p^1 e^{-\left(w_1^1 + x_1^1\right)/\rho} + p^2 e^{-\left(w_1^2 + x_1^2\right)/\rho}}{p^1 e^{-w_1^1/\rho} + p^2 e^{-w_1^2/\rho}} \right)$.

Notice that the period 1 buying price equals the buying price with the payoffs minus the buying price without the payoffs. This leads to the observation that an alternative approach to this problem is to calculate the buying price for two portfolios (one with and one without the payoffs) and the difference between the two is the buying price for the payoffs. For example, take the case when there is negative correlation. The buying price for the portfolio of existing wealth is \$9.6M and the buying price for the price $\$0.6\text{M} < \$1.3\text{M}$). It is important to realize that the large difference in buying prices is due to the correlation, not to the size of the payoffs; e.g., it can be shown that the relative difference in the buying prices is still very dramatic when the project is reduced to a fraction of the size of the original project.

Example 2.3: Insurance Opportunity Using a Logarithmic Utility Function. A decision maker currently has $100,000 and owns a home and wants to find his or her buying price for the purchase of fire insurance. The home can be sold for $100,000 in period 1 but has a 5 percent chance of burning down before period 1 (i.e., $\boldsymbol{p} = [0.05 \quad 0.95]$). The decision maker's utility function is $U(z_0, \boldsymbol{z_1}) = \ln(z_0) + p^1 10\ln(z_1^1) + p^2 10\ln(z_1^2)$, where z is in thousands of dollars. The homeowner can borrow and lend at 10 percent per period and markets are incomplete. Without insurance, it is optimal for the homeowner to spend $16,538 in period 0 and to invest the rest so that he or she will have $91,808 in period 1. Thus, including the value of the house, $\boldsymbol{w_1} = [91.808 \quad 191.808]$.

Rather than performing a full optimization, assume that there is no change in period 0 consumption or period 1 wealth so that the decision maker's total wealth with the net payoff equals $\boldsymbol{\hat{w}_1} = [191.808 - B_1 \quad 191.808 - B_1]$. A fixed-point approach is used to calculate B_1, where $\boldsymbol{\psi_1}(B_1) \cdot \boldsymbol{x_1} = \psi_1^0 B_1$. The state-price vector is $\boldsymbol{\psi_1} = [0.0631 \quad 0.8460]$ and the period 0 buying price is $B_0 = \$6{,}310$; this result is almost identical to a full optimization.

---

portfolio of existing wealth plus the payoffs is $10.9M. Thus, the buying price for the payoffs is $1.3M.

3. Multi-Period Model

Chapter 2 presented a single valuation approach applicable in complete and incomplete markets in a discrete time, single-period setting. This chapter extends the results to a multi-period setting. While this section follows the format from Chapter 2, the way in which uncertainty is revealed over time needs to be determined in a multi-period setting. There are several ways to make this determination. One approach is to use a probability space (Duffie 1992). When the events of the probability space are all taken together, the result is a filtration that represents how information is revealed through time. Another approach is to partition the state space. While the filtration approach has its advantages in continuous time, the partitions approach is taken here because of its intuitive appeal.

This section develops a single valuation approach applicable in complete and incomplete markets in a discrete time, multi-period setting. Assumptions and definitions are presented in Section 3.1, with a particular focus on partitions. Section 3.2 shows that the buying price for any investment satisfies the condition that utility with the net payoffs equals utility without the net payoffs and the condition that utility cannot be improved by changing consumption or market transactions. Section 3.3 introduces the assumption of a time- and state-separable utility function and demonstrates that the decision maker's buying price for an investment is approximately equal to the state-price weighted sum of its payoffs; results are exact when markets are complete or the utility function is additive exponential. Section 3.4 iteratively calculates the buying price using a dynamic programming approach. Exponential and logarithmic examples are included.

3.1 Setting

Definition 3.1: *S* is the set of all possible states that can occur at times *t* equal to 0, 1, …, *T*.[6] Uncertainty is resolved as time progresses so that at time *t*, states that have occurred prior to time *t* are known. This set of states $s_0, s_1, \cdots, s_t$ up to time *t* is denoted as $\{\boldsymbol{s}\}_t$. Thus, $\boldsymbol{S} | \{\boldsymbol{s}\}_t \subset \boldsymbol{S}$ is the set of all possible states that can occur over times *t*+1, …, *T* given that the particular states $s_0, \ldots, s_t$ occurred at times 0, …, *t*.

Definition 3.2: A partition of $\boldsymbol{S}|\{\boldsymbol{s}\}_t$ is a collection of disjoint non-empty subsets of $\boldsymbol{S}|\{\boldsymbol{s}\}_t$ whose union is $\boldsymbol{S}|\{\boldsymbol{s}\}_t$. One way to partition $\boldsymbol{S}|\{\boldsymbol{s}\}_t$ is in terms of the current state and future partitions. Let $s_t|\{\boldsymbol{s}\}_{t-1}$ refer to state s_t at time t given that states $s_0,\ldots,s_{t-1}$ have occurred at times 0, …, t-1; $\{\boldsymbol{s}\}_{t-1}=\{\varnothing\}$ when t equals 0. A partition at time t of $\boldsymbol{S}|\{\boldsymbol{s}\}_t$ for $0\le t\le T$ is the set that contains the current state and the set of all partitions at the next period given that the current state has occurred. That is, the partition $P_t|\{\boldsymbol{s}\}_t=\{s_t|\{\boldsymbol{s}\}_{t-1},\{P_{t+1}|\{\boldsymbol{s}\}_t\}\}$ for $0\le t\le T-1$. Figure 3-1 illustrates what partitions look like when the state space is $\boldsymbol{S}=\{1_0,1_1,2_1,1_2|1_1,2_2|1_1,1_2|2_1,2_2|2_1\}$.[7] The period 0 partition is the largest oval, the period 1 partitions are the medium sized ovals, and the period 2 partitions are the smallest ovals. The period 0 partition contains all subsequent partitions and thus the entire state space since there is no uncertainty in period 0.

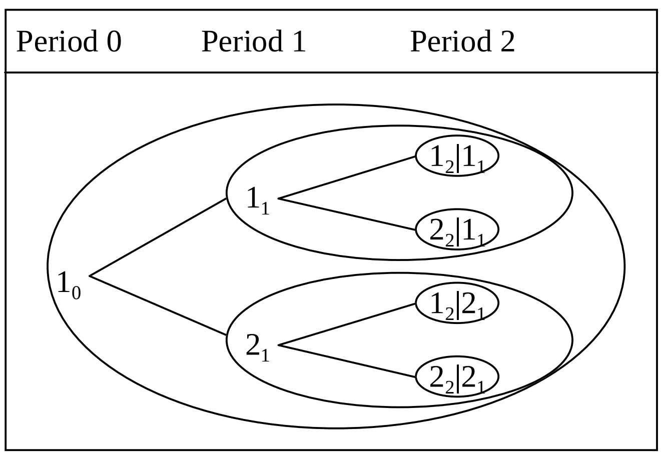

Figure 3-1. Uncertain states (numbers) and partitions (ovals) of $\boldsymbol{S}$.

---

[6] As in Section 2, all variables that have more than one element are presented in bold type.

[7] State 1_0 is not explicitly included in periods 1 and 2 for notational simplicity because it is the same in all states.

Definition 3.3: $S_0 \in R^1$ is the period 0 price of an investment that has time- and state-dependent payoffs $\boldsymbol{x}^{P_t|\{\boldsymbol{s}\}_t} \in R^{S|\{\boldsymbol{s}\}_t}$ in partition $P_t|\{\boldsymbol{s}\}_t$; i.e., $\boldsymbol{x}^{P_t|\{\boldsymbol{s}\}_t} = \left\{\boldsymbol{x}_t^{P_t|\{\boldsymbol{s}\}_t}, \cdots, \boldsymbol{x}_T^{P_t|\{\boldsymbol{s}\}_t}\right\}$.

Definition 3.4: $B_\tau^{P_t|\{\boldsymbol{s}\}_t} \in R$ is the decision maker's period τ buying price for the investment's payoffs of $\boldsymbol{x}^{P_t|\{\boldsymbol{s}\}_t}$. It is the amount that the decision maker is willing to pay in period τ in the partition $P_t|\{\boldsymbol{s}\}_t$ in order to receive the payoff vector $\boldsymbol{x}^{P_t|\{\boldsymbol{s}\}_t}$. It is assumed that $t \le \tau \le T$. That is, $B_\tau^{P_t|\{\boldsymbol{s}\}_t}$ cannot occur before the partition is defined or after the end of the analysis. The definition of $B_\tau^{P_t|\{\boldsymbol{s}\}_t}$ is made more precise in Theorem 3.1. The net payoffs associated with the payoffs $\boldsymbol{x}^{P_t|\{\boldsymbol{s}\}_t}$ and the buying price $B_\tau^{P_t|\{\boldsymbol{s}\}_t}$ are $\left\{\boldsymbol{x}_t^{P_t|\{\boldsymbol{s}\}_t}, \cdots, \boldsymbol{x}_\tau^{P_t|\{\boldsymbol{s}\}_t} - B_\tau^{P_t|\{\boldsymbol{s}\}_t}\mathbf{1}, \cdots, \boldsymbol{x}_T^{P_t|\{\boldsymbol{s}\}_t}\right\}$ where the **1** vector corresponds to the number of states at time τ in partition $P_t|\{\boldsymbol{s}\}_t$.

Definition 3.5: $\Delta\boldsymbol{c} \in R^S$ are the changes in consumption due to re-optimization after the net payoffs are added.

Example 3.1: In order to illustrate the definition of net payoffs, consider the state space from Figure 3-1 with an investment that has period 2 payoffs given that state 2 occurred in period 1. Since all payoffs occur within the partition $P_1|2_1$, the payoffs are $\left\{x_2^{1|2}, x_2^{2|2}\right\}$ and the net payoffs can be written as either $\left\{-B_1^2, x_2^{1|2}, x_2^{2|2}\right\}$ or $\left\{x_2^{1|2} - B_2^2, x_2^{2|2} - B_2^2\right\}$ as illustrated in Figure 3-2; the time subscripts on the states have been dropped for notational simplicity.

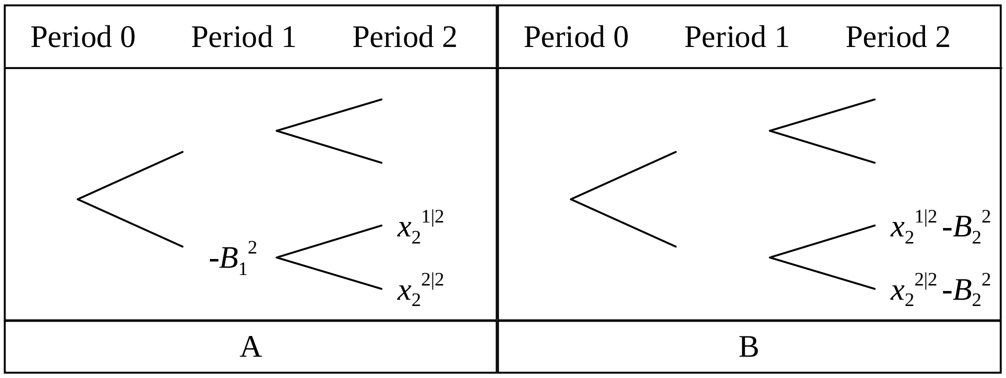

Figure 3-2. Two ways of delineating the net payoffs.

Assumption 3.1: The decision maker has a strictly increasing utility function that maps consumption z over all times and states to a real number. $U\colon R_+^S \to R$ where $U = U(\boldsymbol{z})$ and $\boldsymbol{z} \in R_+^S$. Utility of consumption in the final period can be viewed as utility of wealth. A solution exists to the utility maximization problem, $\max_{\boldsymbol{z}} U(\boldsymbol{z})$, where z is budget-feasible.

Assumption 3.2: $\boldsymbol{c}$ is the solution to the utility maximization problem before the net payoffs are added and $\hat{\boldsymbol{c}}$ is the solution after the net payoffs are added, where $\hat{\boldsymbol{c}}$ is the sum of the original optimal consumption ($\boldsymbol{c}$), the net payoffs $\left\{\boldsymbol{x}_t^{P_t|\{\boldsymbol{s}\}_t}, \cdots, \boldsymbol{x}_\tau^{P_t|\{\boldsymbol{s}\}_t} - B_\tau^{P_t|\{\boldsymbol{s}\}_t}\mathbf{1}, \cdots, \boldsymbol{x}_T^{P_t|\{\boldsymbol{s}\}_t}\right\}$, and the changes in consumption due to re-optimization ($\Delta\boldsymbol{c}$ for $0 \le t \le T$). That is, $\hat{\boldsymbol{c}} = \boldsymbol{c} + \boldsymbol{x} + \Delta\boldsymbol{c}$ except for the case when $t{=}\tau$ and the partition is $P_t|\{\boldsymbol{s}\}_t$ in which case $\hat{\boldsymbol{c}}_\tau^{P_t|\{\boldsymbol{s}\}_t} = \boldsymbol{c}_\tau^{P_t|\{\boldsymbol{s}\}_t} + \boldsymbol{x}_\tau^{P_t|\{\boldsymbol{s}\}_t} - B_\tau^{P_t|\{\boldsymbol{s}\}_t}\mathbf{1} + \Delta\boldsymbol{c}_\tau^{P_t|\{\boldsymbol{s}\}_t}$.

3.2 Necessary Conditions

As in the single-period model from Section 2, two conditions must be satisfied in order for $B_\tau^{P_t|\{\boldsymbol{s}\}_t}$ to be the period τ buying price for the payoffs $\boldsymbol{x}^{P_t|\{\boldsymbol{s}\}_t}$. First, the decision maker's utility with the net payoffs must be the same as the decision maker's utility without the net payoffs. Second, the decision maker cannot improve utility by changing consumption or market transactions. The first condition is developed in Theorem 3.1.

Theorem 3.1: If $B_\tau^{P_t|\{s\}_t}$ is the period τ buying price for the payoffs $\boldsymbol{x}^{P_t|\{s\}_t}$ then the utility of the original consumption plus the net payoffs plus the changes in consumption due to re-optimization equals the utility of the original consumption.

$$U(\hat{\boldsymbol{c}}) = U(\boldsymbol{c}) \tag{3.1}$$

where $\hat{\boldsymbol{c}}$ is defined in Assumption 3.2.

Figure 3-3 illustrates the state by state consumption associated with Theorem 3.1 within the context of Example 3.1. Consumption in the top part of the figure is the original optimal consumption. Consumption in the bottom part of the figure is optimal consumption when the net payoffs are added. The theorem requires that the utility of consumption in the top part of the figure equals the utility of consumption in the bottom part of the figure.

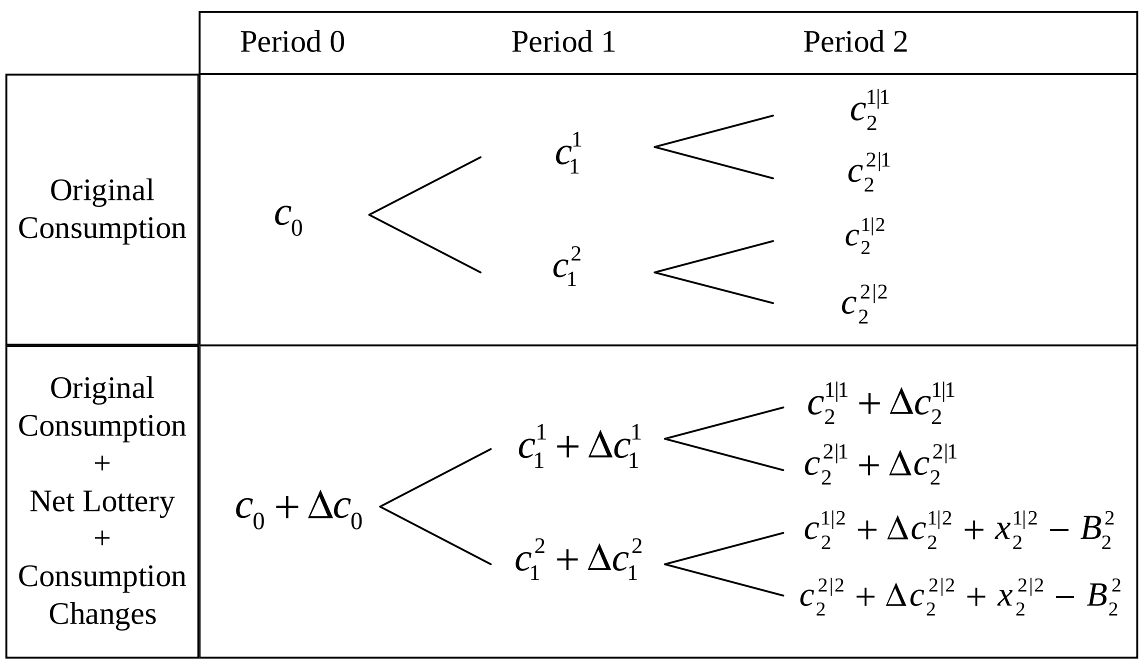

Figure 3-3. Optimal consumption with and without net payoffs.

Optimality Condition: The second condition that must be satisfied is that the decision maker cannot improve utility by changing consumption or market transactions. This condition is true by assumption and has several implications as summarized in the following corollaries; proofs for the corollaries are in the appendix.

Corollary 3.1: The optimality condition implies that, within any given partition $P_t|\{\boldsymbol{s}\}_t$, the marginal utility at the beginning of the partition (i.e., at time *t*) discounted at the risk-free rate between *t* and *t'* in the partition $P_t|\{\boldsymbol{s}\}_t$ ($\psi^0_{t,t'}$) minus the sum of all marginal utilities at time *t'* in the same partition equals zero, where $0 \le t \le t' \le T$.

$$\psi^0_{t,t'} \frac{\partial U}{\partial \hat{c}_t^{\{s\}_t}} - \frac{\partial \boldsymbol{U}}{\partial \hat{\boldsymbol{c}}_{t'}^{P_t|\{s\}_t}} \cdot \mathbf{1} = 0 \tag{3.2}$$

Corollary 3.2: When markets are incomplete, the change in consumption due to re-optimization at the final time *T* is the same across all states in partition $P_{T-1}|\{\boldsymbol{s}\}_{T-1}$ in order to fully satisfy the budget constraint. The result is that any particular budget constraint can be rearranged as follows.

$$\Delta c_T^{s_T|\{s\}_{T-1}} = -\left(\frac{\Delta c_{T-1}^{s_{T-1}|\{s\}_{T-2}}}{\psi^0_{T-1,T}}\right) - \cdots - \left(\frac{\Delta c_2^{s_2|\{s\}_1}}{\psi^0_{2,T}}\right) - \left(\frac{\Delta c_1^{s_1}}{\psi^0_{1,T}}\right) - \left(\frac{\Delta c_0}{\psi_{0,T}}\right) . \tag{3.3}$$

Corollary 3.3: When markets are complete, there is no change in consumption in any state so that $\hat{\boldsymbol{c}} = \boldsymbol{c}$.

3.3 Buying Price

Assumption 3.3: It is assumed for the remainder of this section that the utility function is time- and state-separable.

$$U(\boldsymbol{z}) = \sum_{t=0}^{T} \boldsymbol{u}_\mathbf{t}(\boldsymbol{z}_\mathbf{t}) \cdot \mathbf{1} \tag{3.4}$$

where $\boldsymbol{u}_\mathbf{t}(\boldsymbol{z}_\mathbf{t})$ is the vector of utilities over all states at time *t*.[8]

---

[8] Such a utility function can differentiate the way uncertainty is resolved into the analysis when one models the utility of consumption rather than the utility of income.

Theorem 3.2: For time- and state-separable utility functions, the period τ buying price $B_{\tau}^{P_t|\{s\}_t}$ for payoffs $\boldsymbol{x}^{P_t|\{s\}_t}$ in partition $P_t|\{\boldsymbol{s}\}_t$ is approximately equal to the state-price weighted sum of the payoffs.

$$B_{\tau}^{P_t|\{s\}_t} \cong \sum_{i=t}^{T} \boldsymbol{\psi}_{\tau,i}^{P_t|\{s\}_t} \cdot \boldsymbol{x}_i^{P_t|\{s\}_t} \tag{3.5}$$

where the state prices are $\boldsymbol{\psi}_{\tau,i}^{P_t|\{s\}_t} = \dfrac{\psi_{\tau,i}^{0} \nabla \boldsymbol{u}_{\boldsymbol{i}}^{P_t|\{s\}_t}}{\nabla \boldsymbol{u}_{\boldsymbol{i}}^{P_t|\{s\}_t} \cdot \mathbf{1}}$,

$\nabla u_i^j \left\{ \begin{aligned} &= \frac{u_i^j\left(\hat{c}_i^j\right) - u_i^j\left(c_i^j\right)}{\hat{c}_i^j - c_i^j} && \text{for } \hat{c}_i^j - c_i^j \neq 0 \\ &= u_i^{j\prime}\left(c_i^j\right) && \text{for } \hat{c}_i^j - c_i^j = 0 \end{aligned} \right\}$, and $\psi_{\tau,i}^{0}$ is the risk-free discount factor between τ and i in partition $P_t|\{\boldsymbol{s}\}_t$.

Proof: According to Theorem 3.1 for a time- and state-separable utility function, $B_{\tau}^{P_t|\{s\}_t}$ is the period τ buying price for payoffs $\boldsymbol{x}^{P_t|\{s\}_t}$ if $\sum_{t=0}^{T} \boldsymbol{u}_{\mathbf{t}}(\hat{\boldsymbol{c}}_{\mathbf{t}}) \cdot \mathbf{1} = \sum_{t=0}^{T} \boldsymbol{u}_{\mathbf{t}}(\boldsymbol{c}_{\mathbf{t}}) \cdot \mathbf{1}$. This can be rewritten as $\sum_{t=0}^{T} \nabla \boldsymbol{u}_{\mathbf{t}} \cdot (\hat{\boldsymbol{c}}_{\mathbf{t}} - \boldsymbol{c}_{\mathbf{t}}) = 0$ with ∇u_i^j defined above. Substituting for $\hat{\boldsymbol{c}}_{\mathbf{t}} - \boldsymbol{c}_{\mathbf{t}}$ and rearranging the changes in consumption, the payoffs, and the buying price according to partitions and then dividing by $\nabla \boldsymbol{u}_{\tau}^{P_t|\{s\}_t} \cdot \mathbf{1}$ (because it is strictly positive) and adding $B_{\tau}^{P_t|\{s\}_t}$ results in

$$B_{\tau}^{P_t|\{s\}_t} = \sum_{i=t}^{T} \beta_i \boldsymbol{\psi}_{\tau,i}^{P_t|\{s\}_t} \cdot \boldsymbol{x}_i^{P_t|\{s\}_t} + \left[\frac{1}{\nabla \boldsymbol{u}_{\tau}^{P_t|\{s\}_t} \cdot \mathbf{1}}\right] \left[\begin{aligned} &\nabla u_0 \Delta c_0 \\ &+ \sum_{s_1=1}^{S_1} \Big[\nabla u_1^{s_1} \Delta c_1^{s_1} \\ &\quad + \sum_{s_2|\{s\}_1=1}^{S_2|\{s\}_1} \Big[\nabla u_2^{s_2|\{s\}_1} \Delta c_2^{s_2|\{s\}_1} \\ &\quad\quad + \cdots + \\ &\quad\quad + \sum_{s_T|\{s\}_{T-1}=1}^{S_T|\{s\}_{T-1}} \Big[\nabla u_T^{s_T|\{s\}_{T-1}} \Delta c_T^{s_T|\{s\}_{T-1}} \Big]\Big]\Big] \end{aligned} \right] \tag{3.6}$$

where $\beta_i = \dfrac{\nabla \boldsymbol{u}_i^{P_t|\{s\}_t} \cdot \mathbf{1}}{\psi^0_{\tau,i} \nabla \boldsymbol{u}_\tau^{P_t|\{s\}_t} \cdot \mathbf{1}}$. Theorem 3.2 is proven if $\beta_i \cong 1$ for $t \le i \le T$ and if the second term is approximately equal to zero.

Consider first the β terms. Substituting i for t' in Corollary 3.1 results in $\psi^0_{t,i} \dfrac{\partial U}{\partial \hat{c}_t^{\{s\}_t}} - \dfrac{\partial \boldsymbol{U}}{\partial \hat{\boldsymbol{c}}_i^{P_t|\{s\}_t}} \cdot \mathbf{1} = 0$. Substituting τ for t' in Corollary 3.1 results in $\psi^0_{t,\tau} \dfrac{\partial U}{\partial \hat{c}_t^{\{s\}_t}} - \dfrac{\partial \boldsymbol{U}}{\partial \hat{\boldsymbol{c}}_\tau^{P_t|\{s\}_t}} \cdot \mathbf{1} = 0$. These two equations can be combined to show that $\dfrac{\dfrac{\partial \boldsymbol{U}}{\partial \hat{\boldsymbol{c}}_i^{P_t|\{s\}_t}} \cdot \mathbf{1}}{\psi^0_{\tau,i} \dfrac{\partial \boldsymbol{U}}{\partial \hat{\boldsymbol{c}}_\tau^{P_t|\{s\}_t}} \cdot \mathbf{1}} = 1$. Thus, the β terms are approximately equal to 1 based on conditions of optimality.

Next, consider the bracketed term.

Case 1: Complete Markets. Corollary 3.3 states that $\hat{\boldsymbol{c}} = \boldsymbol{c}$. This results in $\Delta u_i^j = u_i^{j}{}'$ so that the first order conditions from Corollary 3.1 are satisfied exactly and all of the β's equal 1. In addition, the bracketed term is the normalized state-price weighted sum of the changes in consumption. This must equal zero in order to avoid arbitrage opportunities.

Case 2: Incomplete Markets. It can be shown by substituting for each $\Delta c_T^{s_T|\{s\}_{T-1}}$ using Corollary 3.2 that the bracketed term in (3.6) reduces to $\left(\dfrac{1}{\nabla \boldsymbol{u}_\tau^{P_t|\{s\}_t} \cdot \mathbf{1}}\right)\left[\left(\nabla u_0 - \dfrac{\nabla \boldsymbol{u}_T \cdot \mathbf{1}}{\psi^0_{0,T}}\right)\Delta c_0 + \right.$

$\left.\sum_{s_1=1}^{S_1}\left[\left(\nabla u_1^{s_1} - \dfrac{\nabla \boldsymbol{u}_T^{P_1|\{s\}_1} \cdot \mathbf{1}}{\psi^0_{1,T}}\right)\Delta c_1^{s_1} + \cdots + \sum_{s_{T-1}|\{s\}_{T-2}=1}^{S_{T-1}|\{s\}_{T-2}}\left[\left(\nabla u_{T-1}^{s_{T-1}|\{s\}_{T-2}} - \dfrac{\nabla \boldsymbol{u}_T^{P_{T-1}|\{s\}_{T-1}} \cdot \mathbf{1}}{\psi^0_{T-1,T}}\right)\Delta c_{T-1}^{s_{T-1}|\{s\}_{T-2}}\right]\right]\right]$. This term is approximately equal to zero by optimality conditions from Corollary 3.1.

3.4 Buying Price Within a Partition

Determining the period 0 buying price B_0 for payoffs $\boldsymbol{x} \in R^S$ based on the results of Section 3.3 requires that the evaluation be performed over the entire period 0 partition in a single step. This presents a potential weakness because it is not clear how well the respective sets of $\nabla u's$ approximate the optimality conditions from Corollary 3.1. One way to address this issue is to make the following assumption.

Assumption 3.4: The buying price for payoffs in prior time periods does not change with the addition of net payoffs in subsequent time periods.

Example 3.2: Figure 3-4 illustrates the meaning and implications of Assumption 3.4 using the state space from Figure 3-1. Panel A illustrates that the decision maker wants to determine the period 0 buying price B_0 for the period 1 payoffs $\boldsymbol{x} = \left\{\tilde{x}_1^1, \tilde{x}_1^2\right\}$, where $\tilde{x}_1^1 = x_1^1 + \psi_{1,2}^0 B_2^1$ and $\tilde{x}_1^2 = x_1^2 + \psi_{1,2}^0 B_2^2$. Assumption 3.4 states that B_0 is unaffected by the addition of net payoffs in subsequent time periods. Panel B indicates that this allows the decision maker to add the payoffs $\left\{x_2^{1|1}, x_2^{2|1}\right\}$ and $\left\{x_2^{1|2}, x_2^{2|2}\right\}$ and subtract the respective period 2 buying prices B_2^1 and B_2^2 without changing B_0. Panel C shows that the B's in periods 1 and 2 cancel with the result that B_0 is the buying price for either $\left\{x_1^1 + \psi_{1,2}^0 B_2^1, x_1^2 + \psi_{1,2}^0 B_2^2\right\}$ or $\left\{x_1^1, x_1^2, x_2^{1|1}, x_2^{2|1}, x_2^{1|2}, x_2^{2|2}\right\}$.

Assumption 3.4 is so useful because it allows one to iteratively calculate the buying prices of payoff subsets rather than having to calculate the buying price in a single step. One begins with the final date in which there is any uncertainty, calculates the buying price of payoff subsets, adds the result to the payoffs in the previous period, and repeats the process until period 0 is reached.

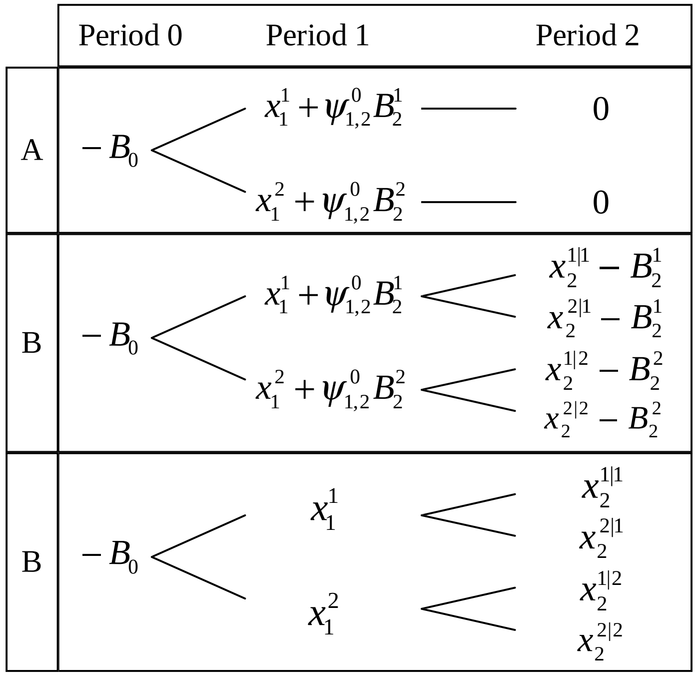

Figure 3-4. Illustration of Assumption 3.4.

More formally, the process is applied as follows. (1) Calculate the time T buying price $B_T^{P_{T-1}|\{\boldsymbol{s}\}_{T-1}}$ for the payoffs $\boldsymbol{x}_T^{P_{T-1}|\{\boldsymbol{s}\}_{T-1}}$ in partition $P_{T-1}|\{\boldsymbol{s}\}_{T-1}$ at time T using (3.5). Repeat this for all $P_{T-1}|\{\boldsymbol{s}\}_{T-1}$ partitions. (2) Discount the results to time T-1 at the risk-free discount rate for a time T-1 buying price of $\psi_{T-1,T}^{0} B_T^{P_{T-1}|\{\boldsymbol{s}\}_{T-1}}$. (3) Add these buying prices to any payoffs for each partition at time T-1 for a result of $\tilde{x}_{T-1}^{P_{T-1}|\{\boldsymbol{s}\}_{T-1}} = x_{T-1}^{P_{T-1}|\{\boldsymbol{s}\}_{T-1}} + \psi_{T-1,T}^{0} B_T^{P_{T-1}|\{\boldsymbol{s}\}_{T-1}}$ for each partition. (4) Convert the utility function of consumption over times T-1 and T to the utility of wealth at time T-1 for each partition. (5) Let T equal T-1 and repeat this process for each stage of uncertainty until T equals 1. The following two examples illustrate how this process is applied for additive exponential and additive logarithmic utility functions when there are no pre-existing uncertain payoffs.[9]

[9] As discussed in Example 2.2, one approach to resolving situations with pre-existing uncertain payoffs for which there are no markets is to calculate the buying price for two portfolios (one with only the pre-existing payoffs and the other with both the pre-existing payoffs as well as the new payoffs). The difference between the two is the buying price for the new payoffs.

Example 3.3: Additive Exponential Utility Function. Suppose that the decision maker's utility function of consumption is exponential from t equals 0 to T: $U(\boldsymbol{z}) = -\sum_{t=0}^{T} k_t e^{-z_t/\rho_t}$. The budget constraint when there are no other uncertainties prior to the addition of the net payoffs is $\sum_{t=0}^{T} \psi_t^0 z_t \le w_0$. The decision maker's problem is to maximize utility subject to the budget constraint.

A dynamic programming approach can be used to reduce the utility of consumption over all periods to the utility of consumption through time T-2 plus the utility of wealth at time T-1 since there is no uncertainty at time T. Specifically, if U_{T-2} is the maximum utility from periods 0 to T-2, the maximized utility equals $U_{T-2} + \max_{z_{T-1}, z_T} \left[-k_{T-1} e^{-z_{T-1}/\rho_{T-1}} - k_T e^{-z_T/\rho_T} \right]$ subject to the budget constraint of $z_{T-1} + \psi_{T-1,T}^0 z_T \le w_{T-1}$ where $\psi_{T-1,T}^0$ is the risk-free discount factor between times T-1 and T. The solution to this problem is $U_{T-2} - A_{T-1} e^{-w_{T-1}/\tilde{\rho}_{T-1}}$ where $\tilde{\rho}_{T-1} = \rho_{T-1} + \psi_{T-1,T}^0 \rho_{T-1}$ and A_{T-1} is a constant. This calculation can be repeated iteratively resulting in a utility function that equals $-\sum_{t=0}^{\tau-1} k_t e^{-c_t/\rho_t} - A_\tau e^{-w_\tau/\tilde{\rho}_\tau}$ where $\tilde{\rho}_\tau = \sum_{t=\tau}^{T} \psi_{\tau,t}^0 \rho_t$ and A_τ is a constant. This approach of calculating a new risk tolerance as one moves backward in time is consistent with results identified by Smith and Nau (1995).

To make the example concrete, assume that there are three periods (0, 1, and 2), that ρ_t equals 1 in each period, that the risk-free discount rate is 5 percent per period, that the payoff is either 1 or 0 in each period with a 50 percent probability, and that wealth is constant. The same formula used for calculating state prices in footnote 5 is applicable here with an adjustment being made to the risk tolerance parameter as the tree is folded back. The result is that $\boldsymbol{\psi}_{1,2}^1 = \boldsymbol{\psi}_{1,2}^2 = [0.362 \quad 0.591]$ so that $B_1^1 = B_1^2 = 0.362 = [0.362 \quad 0.591] \cdot [1 \quad 0]$. Adding these buying prices to the period 1 payoffs results in a new set of payoffs of $\tilde{\boldsymbol{x}}_1 = [1.362 \quad 0.362]$ and a risk tolerance of 1.95.

This results in the state-price vector $\boldsymbol{\psi}_{\mathbf{0,1}} = \begin{bmatrix} 0.416 & 0.537 \end{bmatrix}$ for a period 0 buying price of $0.761 = \begin{bmatrix} 0.416 & 0.537 \end{bmatrix} \cdot \begin{bmatrix} 1.362 & 0.362 \end{bmatrix}$. This result is identical to a full optimization.

Example 3.4: Additive Logarithmic Utility Function. Suppose that the decision maker's utility function of consumption is logarithmic in every period. That is, $U(\boldsymbol{z}) = \sum_{t=0}^{T} k_t \ln(z_t)$. The budget constraint when there are no other uncertainties prior to the addition of the net payoffs is $\sum_{t=0}^{T} \psi_t^0 z_t \le w_0$.

A dynamic programming approach is used to reduce this to the utility of consumption through time *T*-2 plus the utility of wealth at time *T*-1 since there is no uncertainty at time *T*. Specifically, if U_{T-2} is the maximum utility from times 0 to *T*-2, the maximized utility equals $U_{T-2} + \max_{z_{T-1}, z_T} \left[k_{T-1} \ln(z_{T-1}) + k_T \ln(z_T) \right]$ subject to the budget constraint of $z_{T-1} + \psi_{T-1,T}^0 z_T \le w_{T-1}$. The solution to this problem when repeated iteratively is $U(\boldsymbol{z}) = \left[\sum_{t=0}^{\tau-1} k_t \ln(c_t) \right] + \left[\left(\sum_{t=\tau}^{T} k_t \right) \ln(w_\tau) \right] + A_\tau$, where A_τ is a constant.

To make the example concrete, let the discount rate equal 5 percent, initial wealth equals 2.84, and $k_0 = 1, k_1 = 1/\psi_1^0$, and $k_2 = 1/\psi_2^0$. These assumptions result in an initial state of consumption of 1 in all periods.

Now consider payoffs where the payoff will be either C or 0 in each period with equal probability, where C ranges from 0 to 6. The state prices in a given partition $P_t|\{\boldsymbol{s}\}_t$ at time τ are solved for iteratively using the formula $\boldsymbol{\psi}_{\boldsymbol{\tau-1,\tau}}^{P_{\tau-1}|\{s\}_{\tau-1}} \left(B_\tau^{P_{\tau-1}|\{s\}_{\tau-1}} \right) \cdot \boldsymbol{x}_{\mathbf{1}}^{P_{\tau-1}|\{s\}_{\tau-1}} = \psi_{\tau-1,\tau}^0 B_\tau^{P_{\tau-1}|\{s\}_{\tau-1}}$ (as in the single period model, it is assumed that $\Delta \boldsymbol{c} = \mathbf{0}$ since the change in consumption is the same across all states in a given partition at the final time when markets are incomplete). Figure 3-5 presents the period 0 buying price as a percent of initial wealth for the range of payoffs evaluated. The figure suggests that, while not exact, the result is a good approximation of the true buying price obtained by performing a full optimization. The approximation is good because the first-order conditions of Corollary

3.1 are approximately satisfied, not because of a lack of change in consumption when the net payoffs are added.[10]

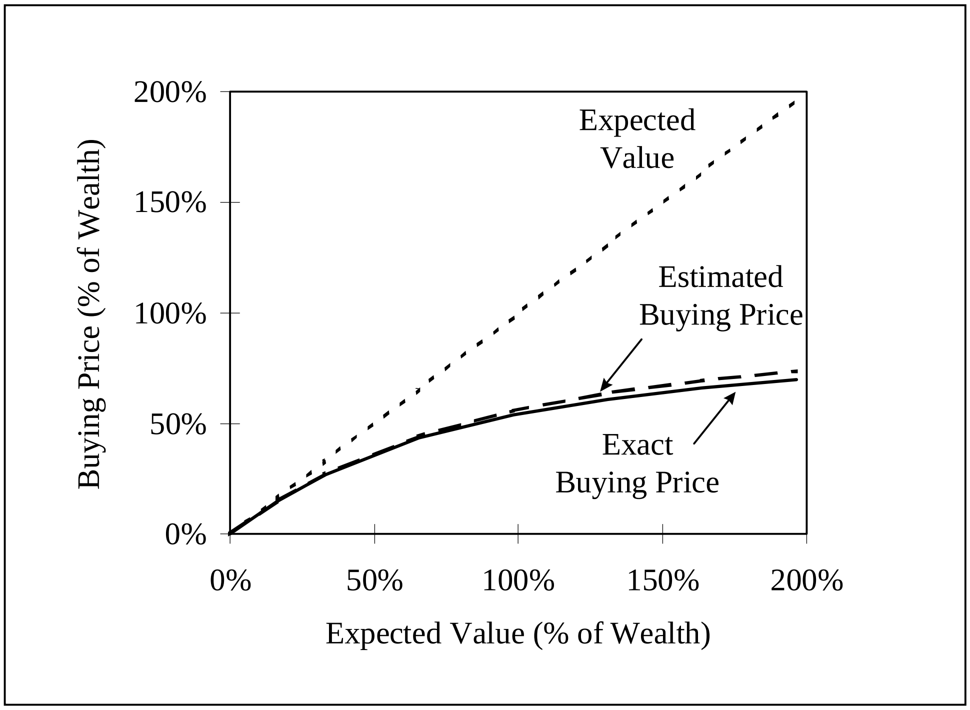

Figure 3-5. Estimated and exact buying price for logarithmic utility function.

[10] To illustrate, assume that the payoff is 6 or 0 in every period. Initial consumption is 1 in every state and period. New state by state consumption is $\{c_0, c_1^1, c_1^2, c_2^{1|1}, c_2^{2|1}, c_2^{|2}, c_2^{2|2}\} = \{0.50, 4.00, 0.27, 8.57, 2.57, 6.14, 0.14\}$.

4. Application: Investment in Distributed Electricity Resources

Electric utilities have become increasingly reluctant to invest in long lead-time, large-scale power generation projects for several reasons. First, facilities have often cost more and taken longer to construct than initially anticipated. Second, actual demand has often not met expectations with the result that the investments were unnecessary. Third, the investments have had a greater financial risk than initially anticipated because: a) regulatory commissions viewed them as imprudent and prevented full cost recovery in some cases; and b) competition from lower cost suppliers prevented full cost recovery in other cases. The result is that generation investments have shifted away from long lead-time, large-scale generation to short lead-time, modular generation. Sacramento Municipal Utility District's experience with nuclear power represents a good case study of some of these issues (Smeloff and Asmus 1997).

Integrated utilities are being restructured into unregulated generation companies and regulated transmission and distribution (T&D) companies. The regulated T&D companies will continue to serve the role of power delivery. Historically, a utility's revenues have been based on its assets. Given that this practice continues, these utilities may be tempted to expand the T&D system whenever there appears to be a need.

Large T&D investments as typified by new transmission facilities are somewhat analogous to large generation system investments: they are long lead-time, large-scale investments. As a result, T&D companies can learn valuable lessons from the generation investment experience and avoid some of the potential problems associated with long lead-time, large-scale investments. The lessons are as follows. First, these investments may cost more and take longer to construct than initially anticipated. Second, a careful evaluation of demand uncertainty is essential prior to committing to any long lead-time investment. Third, an accurate treatment of the financial risk associated with the investment is critical, even within a regulatory environment; this is particularly true when other alternatives represent a competitive threat to the utility.

Distributed resources can enable electric utilities to increase system capacity without some of the drawbacks associated with large-scale, long lead-time transmission and distribution system (T&D) investments. This chapter evaluates the value of flexibility associated with distributed resources to satisfy T&D capacity needs. It calculates investment cost as a function of investment lead-time. The chapter begins with a calculation of the expected cost of distributed resources to illustrate how to perform the analysis for a risk-neutral decision-maker. Section 4.2 applies the results from Chapter 3 to account for risk-attitude under three situations: 1) there is only demand uncertainty; 2) there is demand uncertainty and profit uncertainty, neither of which can be fully hedged by entering into market transactions; and 3) there is demand uncertainty and profit uncertainty, where the profit uncertainty can be hedged (see Hoff 1997a for more details as well as how to incorporate modularity into the analysis.)

4.1 Expected Distributed Generation Cost

Distributed resources typically have shorter construction lead-times than large T&D investments. This provides decision-makers with flexibility regarding when to make investments. This section illustrates how to characterize the dynamic nature of demand uncertainty and the important interaction that occurs between demand uncertainty and investment lead-time.

Utility planners typically incorporate demand uncertainty into an evaluation by projecting high, average, and low demand growth scenarios. The weakness of this approach is that it only gives a static picture of demand uncertainty. That is, there are only three possible paths that demand can follow.

A more accurate way to capture the dynamic nature of demand uncertainty is to recognize that the rate of demand growth can change over time (Hoff and Herig 1997). Figure 4-1 illustrates this point for a T&D system for a hypothetical utility. The heavy solid line in the top part of the figure shows that historical peak demand increased in 1995, remained constant in 1996, and increased in 1997; the heavy dashed line shows that system capacity remained constant during this period. The light lines indicate an assumption that demand will either increase by 2 MW or remain constant in any given year (each outcome has equal chances); the light dashed lines indicate that a capacity investment is required. This results in many more than three possible paths that demand can follow with the number of possible paths increasing over time.

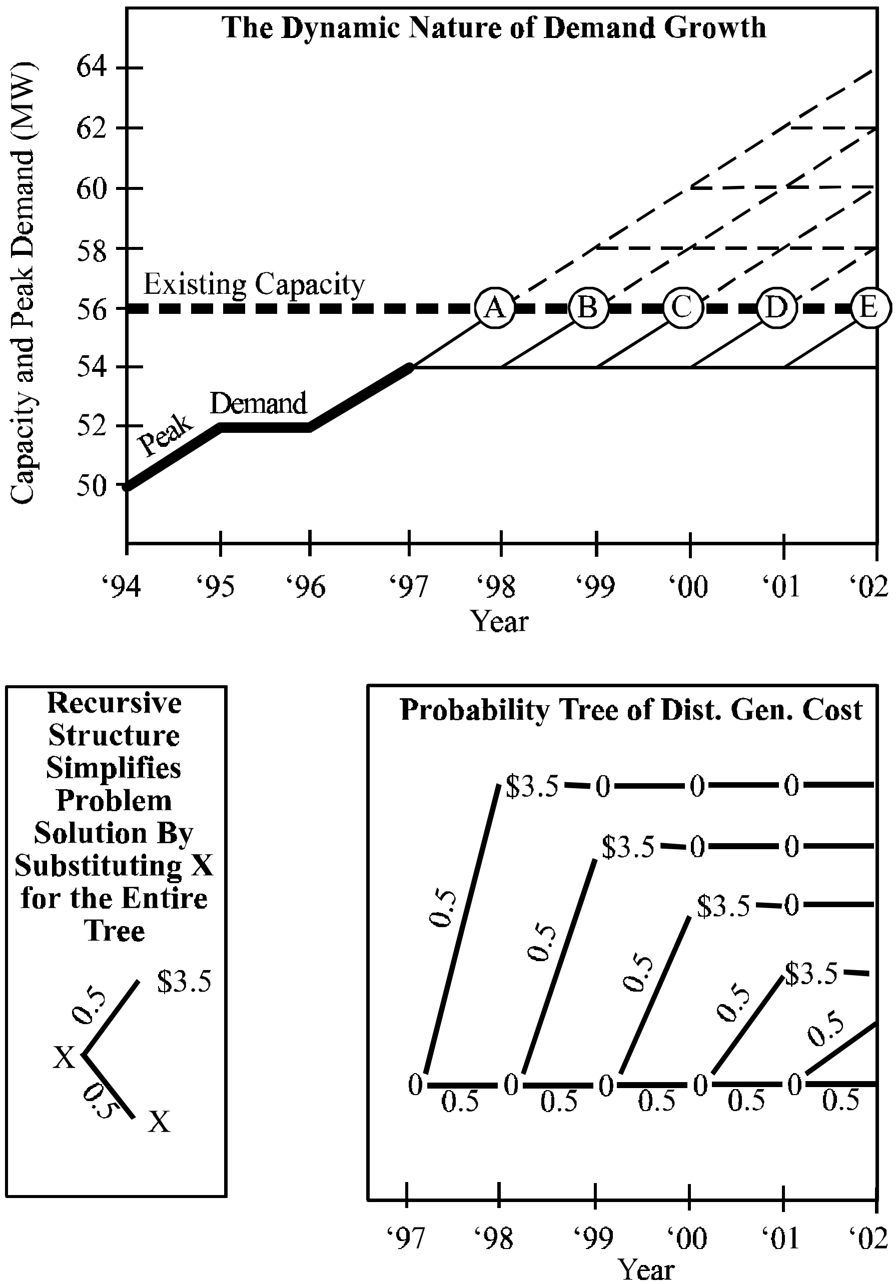

Figure 4-1. Simple example.

The figure indicates that excess capacity will be eliminated if demand increases by 2 MW. The utility has decided that it will increase system capacity by either investing in a system upgrade or by investing in distributed generation. The system upgrade costs $3.000 Million and has a one-year lead-time. The distributed generation costs $3.500 Million and has no lead-time. Both investments provide the same amount of capacity and have infinite lives. The utility has a 10 percent discount rate.

The expected present value cost of the system upgrade is its investment cost of $3.000 Million. This is because the upgrade must begin immediately for the utility to be prepared to satisfy peak demand the first time it occurs.

The expected present value cost of the distributed generation alternative equals the probability that the investment will be made times the investment cost, discounted to the current year. As shown in the figure, the investment could occur in 1998 (point A), 1999 (point B), 2000 (point C), etc. The expected cost of this is calculated using the probability tree in the lower right part of the figure.

While it can be shown that the expected cost of the distributed generation alternative results in a binomial distribution (Hoff 1997b), a simpler solution is to recognize that the tree has a recursive structure (see the lower left part of Figure 4-1). In particular, if demand remains constant in 1998, the probability tree in 1998 is identical to the tree in 1997. The expected present value cost is the probability of high demand growth times the distributed generation cost plus the probability of no growth times the expected present value cost discounted at a rate of 10 percent.

$$\overbrace{X}^{\substack{\text{Expected}\\ \text{Cost}}} = \frac{\overbrace{0.5}^{\substack{\text{Prob. of}\\ \text{High Growth}}} \times \overbrace{\$3.5}^{\substack{\text{Dist. Gen.}\\ \text{Cost}}} + \overbrace{0.5}^{\substack{\text{Prob. of}\\ \text{No Growth}}} \times \overbrace{X}^{\substack{\text{Expected}\\ \text{Cost}}}}{\underbrace{1+10\%}_{\text{Discounting}}} \tag{4.1}$$

A state price interpretation of Equation (4.1), since the decision maker is risk-neutral, is that the state prices are the probabilities discounted at the discount rate.

$$X = \boldsymbol{\psi} \cdot \begin{bmatrix} 3.500 & X \end{bmatrix} \tag{4.2}$$

where $\boldsymbol{\psi} = \begin{bmatrix} 0.4545 & 0.4545 \end{bmatrix}$.

The result is that the expected present value cost for the distributed generation alternative, X, is $2.917 Million, which is less than the $3.000 Million cost of the system upgrade alternative. That is, the distributed generation alternative has a lower expected present value cost than the upgrade.

4.2 Incorporation of Risk Attitiude

The previous section assumed that the decision-maker is risk-neutral. This section incorporates risk attitude into the analysis using the results from Chapter 3. It analyzes the cost of an investment that has no lead-time when there is at least one year until the investment is required. It performs the analysis from three perspectives: (1) there is only demand uncertainty; (2) the firm's value (as represented by its profits) is uncertain in addition to demand but all markets are incomplete; and (3) both value and demand are uncertain with markets being complete for the value uncertainty (i.e., the uncertainty associated with the value of the firm can be hedged by entering into market transactions). It is assumed that the decision maker has an additive exponential utility function and that the risk-free discount rate and the risk-aversion coefficient are constant over time.

4.2.1 Demand Uncertainty

First, consider the situation when demand is the only uncertainty. This results in a similar scenario as a typical decision analysis approach (Howard 1989) with the difference being that there are multiple time periods. It assumes that the firm's only uncertain cash flows are the distributed generation costs and can be represented by the bottom of Figure 4-1. At any particular time, the cost at time τ-1 in state 0 (this is when demand equals 54 MW in Figure 4-1) equals the state price-weighted sum of the costs at time τ in states 0 and 1.

$$X_{\tau-1}^0 = \psi_\tau^1 X_\tau^1 + \psi_\tau^0 X_\tau^0 \tag{4.3}$$

where X_τ^1 is the discounted cost (the negative of the buying price) of all future cash at time τ in state 1 (demand equals 56 MW) and X_τ^0 is the discounted cost of all future cash flows at time τ in state 0 (this is when demand equals 54 MW),

Results from Chapter 3 can be used to show that, after substituting for the state prices when there is an exponential utility function, the equation simplifies as follows.

$$X_{\tau-1}^{0} = \frac{\tilde{\rho}_\tau \ln\left[p\exp\left(\frac{X_\tau^1}{\tilde{\rho}_\tau}\right) + (1-p)\exp\left(\frac{X_\tau^0}{\tilde{\rho}_\tau}\right)\right]}{1+r} \tag{4.4}$$

where the risk-aversion coefficient for all future wealth is the summation of the risk-aversion coefficients over the remaining time periods (i.e., $\tilde{\rho}_\tau = \sum_{t=\tau}^{T} \frac{\rho}{(1+r)^t}$ - see Example 3.3 for a derivation of this result).

The risk-aversion coefficient ($\tilde{\rho}_\tau$) is constant over time since the time frame is infinite so that $\tilde{\rho}_\tau = \frac{\rho(1+r)}{r}$. The discounted cost of all future cash flows at time τ in state 1 is \$3.5 Million (I = \$3.5 Million) because no future investments occur once the investment is made. The recursive structure of the tree can be exploited to find the cost at time τ in state 0. The discounted cash flows in state 0 are identical for all years. In addition, the utility functions are identical up to a scalar multiple. Thus, the cost in state 0 is the same for all times so that $X_\tau^0 = X_{\tau-1}^0$. The subscripts and superscripts are dropped and the appropriate substitutions are made into Equation (4.4). Notice the similarity between Equation (4.5) and Equation (4.1).

$$X = \frac{\tilde{\rho} \ln\left[p\exp\left(\frac{I}{\tilde{\rho}}\right) + (1-p)\exp\left(\frac{X}{\tilde{\rho}}\right)\right]}{1+r} \tag{4.5}$$

A simple optimization program and Equation (4.5) will show that the cost of this investment ranges between \$2.9 Million and \$3.2 Million depending upon the risk-aversion coefficient that is selected when the discount rate is 10 percent. This is consistent with intuition because a risk-neutral decision-maker would pay the expected present value cost of \$2.9 Million (the result from Equation (4.1)) while a highly risk-averse decision-maker would pay \$3.2 Million (he or she would plan for the worst possible scenario even though it only has a 50 percent chance of occurring; the worst case scenario is that the \$3.5 Million cost is incurred for certain in year 1 for a present value cost of \$3.2 Million).

For purposes of illustration, assume that the decision maker has a risk-aversion coefficient of $1 Million (so that $\tilde{\rho} = 11$) and that the discount rate is 10 percent. In this case, the state prices are $\psi = \begin{bmatrix} 0.4605 & 0.4486 \end{bmatrix}$ and the investment cost is $2.923 Million.

4.2.2 Demand Uncertainty and Firm Value Uncertainty (Incomplete Markets)

Next, consider the situation when both demand and the firm's total profits are uncertain and neither uncertainty can be hedged by entering into market transactions. This is the case of incomplete markets. That is, a state-price vector for the firm's value does not exist.

Suppose that the firm's after-tax profits will be high (π^H) or low (π^L) in any year with a probability of q and 1-q respectively. While this model for profits does not allow for large movements in the value of the firm, it will demonstrate the importance of the correlation between distributed generation costs and the firm's profits.

As before, an investment cost of I is incurred when demand increases (probability p) and no future costs are incurred once the investment is made. The correlation between distributed generation costs and firm profits is the same in every year that distributed costs can occur and it equals $corr(I,\pi)$.

Cash flows for the first several years are presented in Figure 4-2. The probabilities for each of the branches are the same as those presented in the lower left part of the figure. The probabilities in the branches are based on the joint probability distribution between p and q, where A is the adjustment in the probability of high demand growth p given q such that the expected value and variance conditions on both distributed generation costs and firm profits are satisfied; $A = corr(I,\pi)\sqrt{p(1-p)q(1-q)}$.[11] V is the buying price of the firm's profits without the distributed generation investment and X is the distributed generation cost.

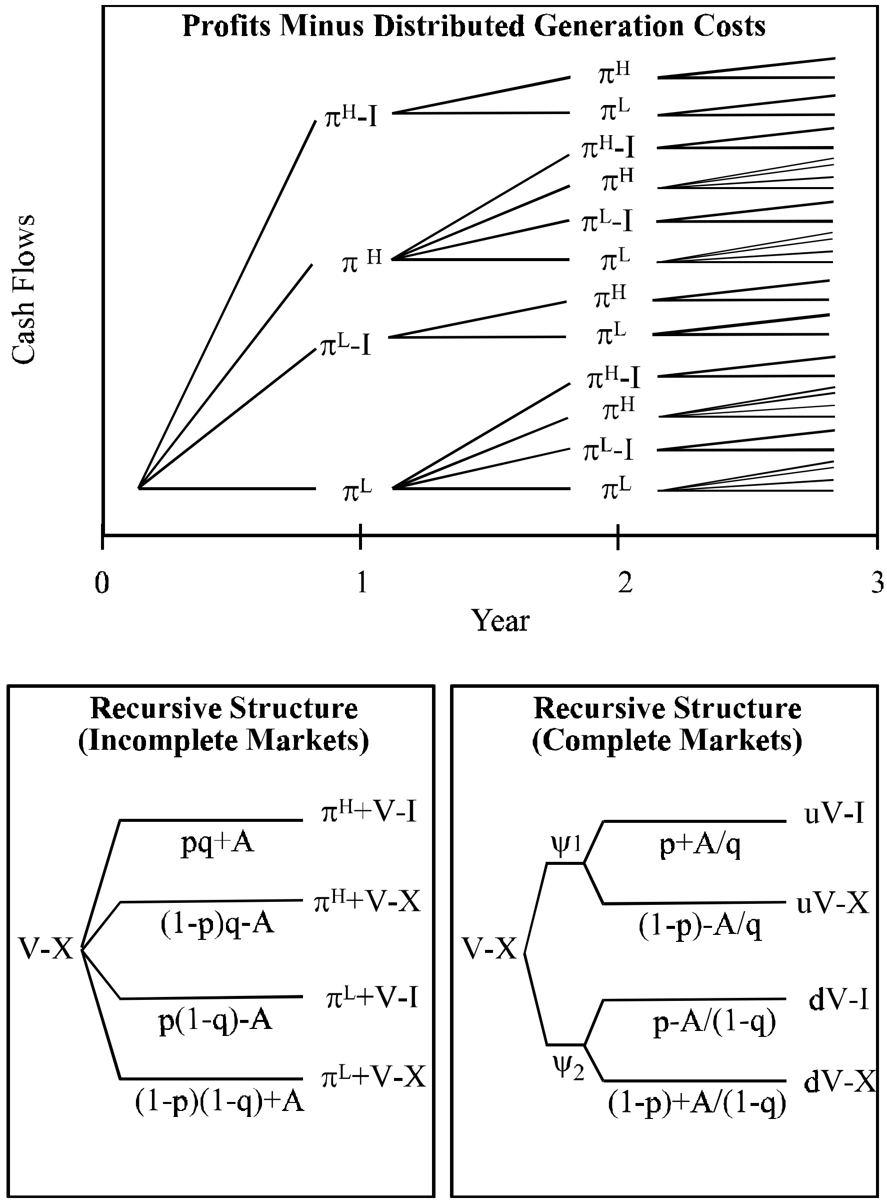

Figure 4-2. Firm's profits and distributed generation costs.

---

[11] The limits on the correlation for a given p and q are $-\min\left(B_L, B_L^{-1}\right) \le corr(I,\pi) \le \min\left(B_U, B_U^{-1}\right)$ where $B_L = \sqrt{\dfrac{pq}{(1-p)(1-q)}}$ and $B_U = \sqrt{\dfrac{p(1-q)}{(1-p)q}}$. This ensures that all probabilities are between 0 and 1. Luenberger (1997, Section 16.4) presents an alternative way to determine the probabilities.

Once again, the recursive structure of the tree (both in terms of cash flows and the utility function) can be exploited. The result is presented in the lower left part of Figure 4-2. As was the case in Equation (4.5), the recursive structure of the tree results in the following formula with the difference being that there are four possible states.

$$V - X = \frac{-\tilde{\rho}\ln\left[\begin{array}{l}(pq + A)\exp\left(-\frac{\pi^H + V - I}{\tilde{\rho}}\right) + ((1-p)q - A)\exp\left(-\frac{\pi^H + V - X}{\tilde{\rho}}\right) + \\ (p(1-q) - A)\exp\left(-\frac{\pi^L + V - I}{\tilde{\rho}}\right) + ((1-p)(1-q) + A)\exp\left(-\frac{\pi^L + V - X}{\tilde{\rho}}\right)\end{array}\right]}{1+r}. \tag{4.6}$$

Equation (4.6) is simplified to find the distributed generation investment cost.

$$X = \frac{\tilde{\rho}\ln\left[(p+P)\exp\left(\frac{I}{\tilde{\rho}}\right) + (1-p-P)\exp\left(\frac{X}{\tilde{\rho}}\right)\right]}{1+r} \tag{4.7}$$

where $\quad P = corr(I,\pi)\sqrt{p(1-p)q(1-q)}\exp\left(\frac{rV}{\tilde{\rho}}\right)\left[\exp\left(\frac{-\pi^H}{\tilde{\rho}}\right) - \exp\left(\frac{-\pi^L}{\tilde{\rho}}\right)\right]$ and

$$V = \frac{-\tilde{\rho}\ln\left[q\exp\left(-\pi^H/\tilde{\rho}\right) + (1-q)\exp\left(-\pi^L/\tilde{\rho}\right)\right]}{r}.$$

The difference between Equation (4.7) and Equation (4.5) is that the actual probability p is replaced the probability $p+P$. This, however, can result in a vastly different solution. This is because P can be viewed as a mathematical adjustment in the probabilities based on the correlation between profits and distributed generation costs as well as the decision maker's risk preferences.

In addition to the assumptions that a distributed generation cost of \$3.5 Million is incurred when demand increases (with probability p=0.5), that the risk-aversion coefficient in any year is \$1 Million (this implies that $\tilde{\rho} = 11$), and the discount rate is 10 percent, assume that the firm's profits will be either \$25 or \$50 Million in any given year (with a probability q=0.5). Equation (4.7) results in a cost of \$1.754 Million, \$2.923 Million, or \$3.153 Million, depending upon whether costs are perfectly correlated, not correlated, or perfectly negatively correlated with profits. Costs that are highly correlated with profits act as a hedge against uncertainty because the costs are incurred only when profits are high (P = -0.4), costs that are uncorrelated reduce to the situation in Section 4.2.1 (P = 0), and costs that are negatively correlated with profits intensify the negative aspects of uncertainty because costs are incurred only when profits are low (P = 0.4). The dependence of the expected present value cost on the correlation with profits becomes even greater as the decision-maker becomes more risk averse.

The four state prices in any period can be reduced to two state prices since there are only two states of demand growth (the two states are: high demand growth/high profits plus high demand growth/low profits; no demand growth/high profits plus no demand growth/low profits). The state prices in the perfectly correlated and perfectly negatively correlated cases are $\psi = [0.0913 \quad 0.8178]$ and $\psi = [0.8260 \quad 0.0830]$ respectively.

4.2.3 Demand Uncertainty and Firm Value Uncertainty (Partially Complete Markets)

Finally, consider the situation when both demand and the firm's value are uncertain and the uncertainty associated with the firm's value can be hedged. This is the case of partially complete markets. Assume that the firm's value follows geometric Brownian motion. This assumption implies that the state prices are constant over time. A diagram of this situation is similar to the top of Figure 4-2 with the difference being that the profits can change over time.

The formulation of the problem is similar to the formulation in Equation (4.6). The primary difference is that the distributed generation cost in each of the demand states is determined and then the result is adjusted for uncertainty and discounting using the appropriate market-based state prices. The recursive structure of the tree at time 0 is shown in the lower right part of Figure 4-2.

$$V - X = \begin{array}{l} \psi^1 \tilde{\rho} \ln\left[\left(p + \dfrac{A}{q}\right)\exp\left(-\dfrac{uV - I}{\tilde{\rho}}\right) + \left((1-p) - \dfrac{A}{q}\right)\exp\left(-\dfrac{uV - X}{\tilde{\rho}}\right)\right] + \\ \psi^2 \tilde{\rho} \ln\left[\left(p - \dfrac{A}{1-q}\right)\exp\left(-\dfrac{dV - I}{\tilde{\rho}}\right) + \left((1-p) + \dfrac{A}{1-q}\right)\exp\left(-\dfrac{dV - X}{\tilde{\rho}}\right)\right] \end{array} \tag{4.8}$$

Equation (4.8) can be simplified to:

$$X = \begin{array}{l} \psi^1 \tilde{\rho} \ln\left[\left(p+\frac{A}{q}\right)\exp\left(\frac{I}{\tilde{\rho}}\right)+\left((1-p)-\frac{A}{q}\right)\exp\left(\frac{X}{\tilde{\rho}}\right)\right]+ \\ \psi^2 \tilde{\rho} \ln\left[\left(p-\frac{A}{1-q}\right)\exp\left(\frac{I}{\tilde{\rho}}\right)+\left((1-p)+\frac{A}{1-q}\right)\exp\left(\frac{X}{\tilde{\rho}}\right)\right] \end{array} \quad (4.9)$$

It is important to note that there is a different lower bound on the cost when markets are complete for the value of the firm. While Equation (4.7) can result in a cost that is very small when distributed generation costs are highly correlated with the firm's value and the decision-maker is highly risk averse, the same is not the case here.

To illustrate, assume that the first state price is smaller than the second state price. Equation (4.9) is minimized when p equals q, the correlation is 1, and $\tilde{\rho}$ is large so that the decision-maker is almost risk-neutral (this is the opposite of the situation in the previous section where a small $\tilde{\rho}$ reduces the cost). Equation (4.9) can be simplified and then solved for X to result in $X = \frac{\psi^1}{1-\psi^2} I$. That is, the lower bound on the distributed generation cost depends on the state prices and thus on the market-traded value of the firm. This suggests that the firm has less exposure when it can hedge its profit risks by entering into market transactions.

4.3 Comparison of Results

Table 1 compares the results from the sections on expected cost, demand uncertainty only , and demand and profit uncertainty when markets are incomplete (Sections 4.1, 4.2.1, and 4.2.2). The most surprising result in the table is that, assuming that the discount rate is the same in all cases, the risk neutral decision maker does not have the lowest cost. Rather, the lowest cost occurs for a risk-averse decision maker when there is a positive correlation between profits and costs. This is because costs that are highly correlated with profits act as a hedge against uncertainty because the costs are incurred only when profits are high.

Table 1. Comparison of Results

	Investment Cost (Millions)	State Prices	
		$3.5M Cost Incurred	$3.5M Cost Not Incurred
Risk Neutral	$2.917	0.4545	0.4545
Risk Averse			
Demand Uncertainty Only	$2.923	0.4605	0.4486
Demand and Profit Uncertainty (Positively Correlated)	$1.754	0.0913	0.8178
Demand and Profit Uncertainty (Negatively Correlated)	$3.153	0.8260	0.0830

5. Conclusions and Future Research

This research presented an approach to value investments with uncertain payoffs that is applicable whether markets are complete or incomplete. The problem was framed within the context of a discrete-time, discrete-states setting. Results suggest that, given a time- and state-separable utility function, a decision maker's buying price for an investment is approximately equal to the state-price weighted sum of its future payoffs; results are exact when markets are complete or the utility function is exponential.

The approach is appealing from a variety perspectives. From a finance perspective, it produces results that are consistent with financial economics when markets are complete (e.g., Duffie 1992) but it is also applicable when markets are incomplete; state prices in incomplete markets have a similar definition as state prices in complete markets. From a decision analysis perspective, it is complementary with other incomplete market results (e.g., a certain equivalent approach by Smith and Nau 1995 and a "Portfolio Decision Analysis" approach by Borison 1996), it extends to problems when the utility function is exponential but there are pre-existing uncertain payoffs,[12] and it extends to non-exponential utility functions.[13] From a real options perspective, it provides a way to calculate "risk-neutral" probabilities when markets are incomplete and it highlights the importance of the interaction between options in a new project with pre-existing options in the portfolio in addition to interactions of options within the same project (Trigeorgis 1996). The approach should appeal to economists because it is approximately marginal utility based pricing. It should also have intuitive appeal to non-economists because one adjusts for preferences in the state prices by giving a heavier weight to the less desirable outcomes. The approach may have computational advantages because the state prices can be estimated without solving a full utility maximization problem for each new investment that is evaluated.

There are several avenues of future research to pursue. From a theoretical perspective, results from this work need to be further generalized. First, the results should be extended to include cases where markets are incomplete but there are more market-traded securities

---

[12] See Example 2.2.

available than the risk-free asset. Second, there may be multiple approaches to calculating state prices when markets are incomplete in the same way that there are multiple approaches to calculating state prices when markets are complete (i.e., the three approaches are: absence of arbitrage, general equilibrium, and single agent optimality); in particular, there may be an approach that does not necessitate the use of a utility function. Third, the results can be generalized to include utility functions that are not time- and state-separable. Fourth, the results can be employed within a game theoretic framework to make investment decisions when there is both competition and uncertainty.[14] Fifth, the results can be developed in a continuous time setting.[15]

From an empirical perspective, the usefulness of the results of this research could be increased. This will be accomplished in a variety of ways. First, a set of guidelines should be developed to bound the level of error associated with non-exponential utility functions in incomplete markets. This research suggests that the error is small but it does not explicitly identify how small. Second, there may be computational advantages with this approach because the sate prices are estimated without solving a full utility maximization problem for each new investment that is evaluated. These advantages should be verified. Third, the theory needs to be applied to real problems to demonstrate its applicability; a promising area is the field of real options. Problems with the option to wait, to invest incrementally, to expand or contract operating scale, to abandon investment, to adjust inputs or outputs, or to have future growth opportunities should be evaluated. While these types of problems occur within a wide range of industries, the use of this research in evaluating future growth options will be of particular interest to groups such as venture capitalists and R&D departments because they face high uncertainty and have incomplete markets. Finally, the results need to be simplified and then communicated in a manner that facilitates their usefulness to a broad range of decision makers.

---

[13] See Example 2.3 and Example 3.4.
[14] This area of research was indentified by Kevin Zhu.
[15] These results are already well established when markets are complete.

6. Appendix: Proofs

Proof for Corollary 2.1: The optimality condition, the fact that the utility function is strictly increasing, and the decision maker's ability to purchase any size increments of $(\Delta c_0, \Delta \boldsymbol{w}_1)$ subject to budget constraints imply that $U(\hat{c}_0, \hat{\boldsymbol{w}}_1)$ is the solution to the problem $\max_{\Delta c_0, \Delta \boldsymbol{w}_1 \in Y} U(\hat{c}_0, \hat{\boldsymbol{w}}_1)$ where Y is the budget-feasible set. Consider two cases.

Case 1: Incomplete Markets. There is no opportunity to adjust period 1 consumption state by state when markets are incomplete and there is only the opportunity for risk-free borrowing and lending. Thus, $\Delta \boldsymbol{w}_1 = [\Delta w_1 \quad \Delta w_1 \quad \ldots \quad \Delta w_1]$ with $\Delta w_1 \in R^1$ and the budget constraint is $\Delta c_0 + \psi_1^0 \Delta w_1 \leq 0$. The first order condition for optimality results in $\frac{\partial U}{\partial \hat{c}_0} = \lambda$ and $\frac{\partial \boldsymbol{U}}{\partial \hat{\boldsymbol{w}}_1} \cdot \mathbf{1} = \psi_1^0 \lambda$. These two equations are solved for λ and combined to result in Corollary 2.1.

Case 2: Complete Markets. There is the opportunity to adjust period 1 consumption state by state when markets are complete. Thus, $\Delta \boldsymbol{w}_1 = \left[\Delta w_1^1 \quad \Delta w_1^2 \quad \ldots \quad \Delta w_1^S\right]$ and the budget constraint is $\Delta c_0 + \boldsymbol{\psi}_1 \cdot \Delta \boldsymbol{w}_1 \leq 0$ where the state-price vector is $\boldsymbol{\psi}_1 \in R_{++}^S$ with $\boldsymbol{\psi}_1 = \left[\psi_1^1 \quad \psi_1^2 \quad \ldots \quad \psi_1^S\right]$. The first order condition for optimality results in $\frac{\partial U}{\partial \hat{c}_0} = \lambda$ and $\frac{\partial U}{\partial \hat{w}_1^i} = \psi_1^i \lambda$ for $1 \leq i \leq S$. Summing over all i, $\frac{\partial \boldsymbol{U}}{\partial \hat{\boldsymbol{w}}_1} \cdot \mathbf{1} = \boldsymbol{\psi}_1 \lambda \cdot \mathbf{1} = \psi_1^0 \lambda$ since the price of receiving one unit of consumption in every state in the future is equal to the price of a risk-free asset (i.e., $\boldsymbol{\psi}_1 \cdot \mathbf{1} = \psi_1^0$). These two equations are solved for λ and combined to result in Corollary 2.1.

Proof for Corollary 2.2: The budget constraint when markets are incomplete is that $\Delta c_0 + \psi_1^0 \Delta w_1 \leq 0$. This budget constraint is satisfied with equality since the utility function is strictly increasing so that $\Delta w_1 = -\Delta c_0 / \psi_1^0$.

Proof for Corollary 2.3: When markets are complete, period 1 wealth is changed state by state by re-optimizing so as to exactly offset the change in wealth due to the net payoff.

This satisfies Theorem 2.1, since $U(c_0 + 0, \boldsymbol{w}_1 + \boldsymbol{0}) = U(c_0, \boldsymbol{w}_1)$, and the optimality condition, since $U(c_0, \boldsymbol{w}_1)$ was maximized prior to the addition of the net payoffs.

Proof for Corollary 2.4: Assume that $\Delta c_0 = 0$. Corollary 2.1 requires that $\psi_1^0 u_0'(c_0) = \boldsymbol{u}_1'(\boldsymbol{w}_1) \cdot \boldsymbol{1}$ for the initial condition and $\psi_1^0 u_0'(c_0) = \boldsymbol{u}_1'(\hat{\boldsymbol{w}}_1) \cdot \boldsymbol{1}$ with the net payoffs so that $\boldsymbol{u}_1'(\boldsymbol{w}_1) \cdot \boldsymbol{1} = \boldsymbol{u}_1'(\hat{\boldsymbol{w}}_1) \cdot \boldsymbol{1}$. Theorem 2.1 requires that $\boldsymbol{u}_1(\boldsymbol{w}_1) \cdot \boldsymbol{1} = \boldsymbol{u}_1(\hat{\boldsymbol{w}}_1) \cdot \boldsymbol{1}$. These two equations are satisfied simultaneously only when the utility function is linear or exponential. Conversely, assume that $U(z_0, \boldsymbol{z}_1) = u_0(z_0) + \boldsymbol{u}_1(\boldsymbol{z}_1) \cdot \boldsymbol{1}$, where $u_0(z_0)$ is arbitrary and $\boldsymbol{u}_1(\boldsymbol{z}_1) \cdot \boldsymbol{1} = -\sum_{i=1}^{S} \alpha^i e^{-z_1^i/\rho}$. Corollary 2.1 requires that $\psi_1^0 u_0'(c_0) = (-1/\rho)\boldsymbol{u}_1(\boldsymbol{w}_1) \cdot \boldsymbol{1}$ for the initial condition. Multiplying by -ρ and adding $u_0(c_0)$ results in $u_0(c_0) - \rho\psi_1^0 u_0'(c_0) = U(c_0, \boldsymbol{w}_1)$. Likewise, $u_0(\hat{c}_0) - \rho\psi_1^0 u_0'(\hat{c}_0) = U(\hat{c}_0, \hat{\boldsymbol{w}}_1)$ for the net payoff. Combining these two equations using Theorem 2.1 results in $u_0(c_0) - \rho\psi_1^0 u_0'(c_0) = u_0(\hat{c}_0) - \rho\psi_1^0 u_0'(\hat{c}_0)$. c_0 must equal $\hat{c}_0$ since $u_0(z_0)$ has an arbitrary form so that Δc_0 equals 0.

Proof for Corollary 3.1: Two cases need to be considered: incomplete markets and complete markets. The objective function in both cases is to maximize utility subject to budget constraints; i.e., $\max_{\Delta \boldsymbol{c} \in Y} U(\hat{\boldsymbol{c}})$ where Y is the budget-feasible set. The difference between the two cases comes in the budget-feasible set Y. When markets are complete, there is a state price for every time and state ($\boldsymbol{\psi} \in R_{++}^{S}$) so that the only budget constraint is $\Delta \boldsymbol{c} \cdot \boldsymbol{\psi} \leq 0$.

In contrast, there are multiple budget constraints when markets are incomplete. In fact, there are as many budget constraints as there are possible combinations of $\Delta\boldsymbol{c}$'s. The number of possible combinations of $\Delta\boldsymbol{c}$'s equals the number of unique branches at time T. Each constraint is $\Delta \boldsymbol{c} | \{\boldsymbol{s}\}_T \cdot \boldsymbol{\psi}^0 \leq 0$ where $\Delta \boldsymbol{c} | \{\boldsymbol{s}\}_T \in R^{T+1}$ is the change in consumption when the path $\{\boldsymbol{s}\}_T$ is taken from period 0 to T and $\boldsymbol{\psi}^0 \in R_{++}^{T+1}$ is the vector of risk-free discount factors for each period.

Case 1: Incomplete Markets. Select a partition at time *t* of $P_t|\{\boldsymbol{s}\}_t$. Given that all constraints are binding (which is true based on Assumption 3.1), the first order condition for optimality at time *t* is that the marginal utility of consumption plus the change in wealth in state $\{\boldsymbol{s}\}_t$ equals the risk-free rate at time *t* multiplied by the sum of the lagrange multipliers associated with all constraints in partition $P_t|\{\boldsymbol{s}\}_t$. That is, $\frac{\partial U}{\partial \hat{c}_t^{\{s\}_t}} = \psi_t^0 \sum_{P_t|\{s\}_t} \lambda^i$. Similarly, at time $t' \geq t$ in partition $P_t|\{\boldsymbol{s}\}_t$, the marginal utility in state $\{\boldsymbol{s}\}_{t'}$ given state $\{\boldsymbol{s}\}_t$ equals the risk-free rate at time *t'* multiplied by the sum of the lagrange multipliers associated with all constraints in the "sub-partition" $P_{t'}|\{\boldsymbol{s}\}_{t'} \subset P_t|\{\boldsymbol{s}\}_t$. That is, $\frac{\partial U}{\partial \hat{c}_t^{\{s\}_{t'}|\{s\}_t}} = \psi_{t'}^0 \sum_{P_{t'}|\{s\}_{t'},\{s\}_t} \lambda^i$. Since this is true for all sub-partitions $P_{t'}|\{\boldsymbol{s}\}_{t'}$, the results can be summed so that $\frac{\partial \boldsymbol{U}}{\partial \hat{\boldsymbol{c}}_{t'}^{P_t|\{s\}_t}} \cdot \mathbf{1} = \psi_{t'}^0 \sum_{P_t|\{s\}_t} \lambda^i$. Combining the first order conditions from time *t* and time *t'* result in Corollary 3.1.

Case 2: Complete Markets. There is the opportunity to adjust consumption state by state when markets are complete. Select a partition at time *t* of $P_t|\{\boldsymbol{s}\}_t$. The first order condition for optimality at time *t* is $\frac{\partial U}{\partial \hat{c}_t^{\{s\}_t}} = \psi_t^{\{s\}_t} \lambda$. Likewise, the first order condition for optimality at time *t'* for a particular state in partition $P_t|\{\boldsymbol{s}\}_t$ is $\frac{\partial U}{\partial \hat{c}_{t'}^{\{s\}_{t'}|\{s\}_t}} = \psi_{t'}^{\{s\}_{t'}|\{s\}_t} \lambda$. The sum over all states at time *t'* in partition $P_t|\{\boldsymbol{s}\}_t$ is $\frac{\partial \boldsymbol{U}}{\partial \hat{\boldsymbol{c}}_{t'}^{P_t|\{s\}_t}} \cdot \mathbf{1} = \lambda \boldsymbol{\psi}_{t'}^{P_t|\{s\}_t} \cdot \mathbf{1}$. The right hand side of the equation equals the risk-free rate between times *t* and *t'* in partition $P_t|\{\boldsymbol{s}\}_t$ multiplied by the state price at time *t* given $\{\boldsymbol{s}\}_t$. That is, $\boldsymbol{\psi}_{t'}^{P_t|\{s\}_t} \cdot \mathbf{1} = \psi_{t,t'}^0 \psi_t^{\{s\}_t}$. Substituting for this and then combining conditions at time *t* and *t'* results in Corollary 3.1.

Proof for Corollary 3.2: When markets are incomplete, there is a budget constraint for each path taken. The discounted change in consumption for any path $\{\boldsymbol{s}\}_T$ that is taken must equal zero, so that

$\Delta c_0 + \psi_{0,1}^0 \Delta c_1^{s_1} + \psi_{0,2}^0 \Delta c_2^{s_2|\{s\}_1} + \cdots + \psi_{0,T-1}^0 \Delta c_{T-1}^{s_{T-1}|\{s\}_{T-2}} + \psi_{0,T}^0 \Delta c_T^{s_T|\{s\}_{T-1}} = 0$ since the decision maker only has the opportunity to borrow or lend at the risk-free discount rate in incomplete markets. This rearranges to $\Delta c_T^{s_T|\{s\}_{T-1}} = -\left(\frac{\Delta c_{T-1}^{s_{T-1}|\{s\}_{T-2}}}{\psi_{T-1,T}^0}\right) - \cdots - \left(\frac{\Delta c_2^{s_2|\{s\}_1}}{\psi_{2,T}^0}\right) - \left(\frac{\Delta c_1^{s_1}}{\psi_{1,T}^0}\right) - \left(\frac{\Delta c_0}{\psi_{0,T}}\right)$ where $\psi_{i,T}^0$ is the risk-free discount rate between periods i and T. The right hand side is independent of the final state s_T since the change in consumption in the final state must be constant given the previous state in order to fully satisfy the budget constraint.

Proof for Corollary 3.3: When markets are complete, period t consumption is changed state by state so that the changes in consumption due to the net payoffs are exactly offset. This satisfies Theorem 3.1, since $U(\boldsymbol{c} + \boldsymbol{0}) = U(\boldsymbol{c})$, and the optimality condition since $U(\boldsymbol{c})$ was maximized without the net payoffs.

7. Appendix: Partitions and Time- and State-Separable Utility Functions

One issue associated with the results in this research is whether a time- and state-separable utility function is capable of capturing the information about the resolution of uncertainty in a multi-period setting. This appendix demonstrates that it is. This claim is substantiated through the use of an example based on an expected utility analysis.

A decision maker is offered a project that has equal chances of paying \$1.0 Million or \$0.3 Million in period 1 and in period 2. The decision maker has an expected utility function and has \$0 of current wealth. Figure 7-1 illustrates three ways that the time resolution of the uncertainty associated with these payoffs can be interpreted: (A) all uncertainty is resolved immediately; (B) all uncertainty is resolved in period 1; and (C) the uncertainty is resolved sequentially in period 1 and in period 2. The ovals correspond to the payoffs and the numbers above the lines correspond to the probability of the payoffs occurring for these three interpretations.

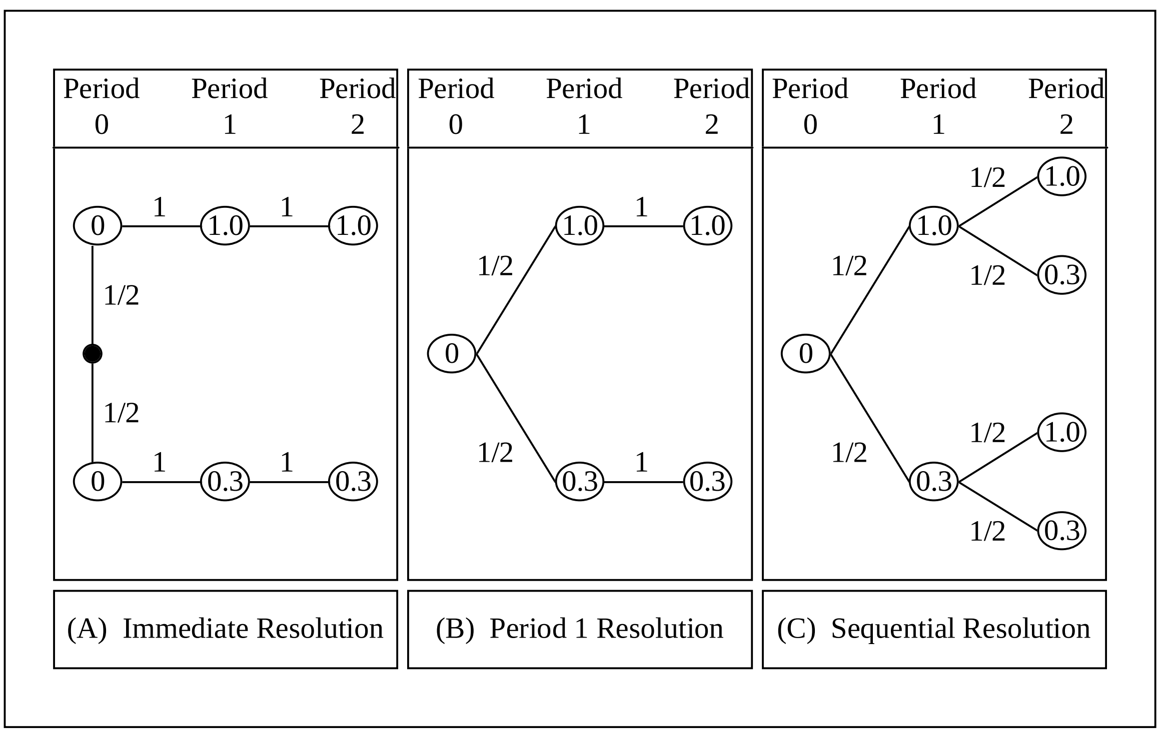

Figure 7-1. Project payoffs and resolution of uncertainty with expected utility.

The figure makes it apparent that the state spaces differ in each of the three case. Since there can be uncertainty in period 0, the dependence of future states on the outcome of period 0 is made explicit only where necessary (i.e., in case A). The state spaces are $\boldsymbol{S_A} = \{1_0, 2_0, 1_1|1_0, 2_1|2_0, 1_2|1_0,1_1, 2_2|2_0,2_1\}$, $\boldsymbol{S_B} = \{1_0, 1_1, 2_1, 1_2|1_1, 2_2|2_1\}$ and $\boldsymbol{S_A} = \{1_0, 1_1, 2_1, 1_2|1_1, 2_2|1_1, 1_2|2_1, 2_2|2_1\}$. Now the question is whether or not a time- and state-separable utility function captures this difference.

If one calculates the utility of the ***payoffs*** of these three formulations, then the utility of A is $U_A = \frac{1}{2}\left[u_0(0) + u_1(1.0) + u_2(1.0)\right] + \frac{1}{2}\left[u_0(0) + u_1(0.3) + u_2(0.3)\right]$, the utility of B is $U_B = u_0(0) + \frac{1}{2}\left[u_1(1.0) + u_2(1.0)\right] + \frac{1}{2}\left[u_1(0.3) + u_2(0.3)\right]$ and the utility of C is $U_C = u_0(0) + \frac{1}{2}\left[u_1(1.0) + \frac{1}{2}\left[u_2(1.0) + u_2(0.3)\right]\right] + \frac{1}{2}\left[u_1(0.3) + \frac{1}{2}\left[u_2(1.0) + u_2(0.3)\right]\right]$. This approach results in identical utilities independent of the form of the utility function (i.e., $U_A = U_B = U_C$).

The problem with such an approach, however, is that it evaluates the utility of the payoffs not the utility of consumption. Specifically, it fails to recognize a decision maker's opportunity to optimize inter-temporal consumption by borrowing and lending. The evaluation should be based on ***consumption*** as depicted in Figure 7-2, where the *c*'s in A, B, and C are not necessarily the same. Subscripts refer to time and superscripts refer to states.[16]

[16] The time subscripts on the state superscripts are omitted for notation simplicity.

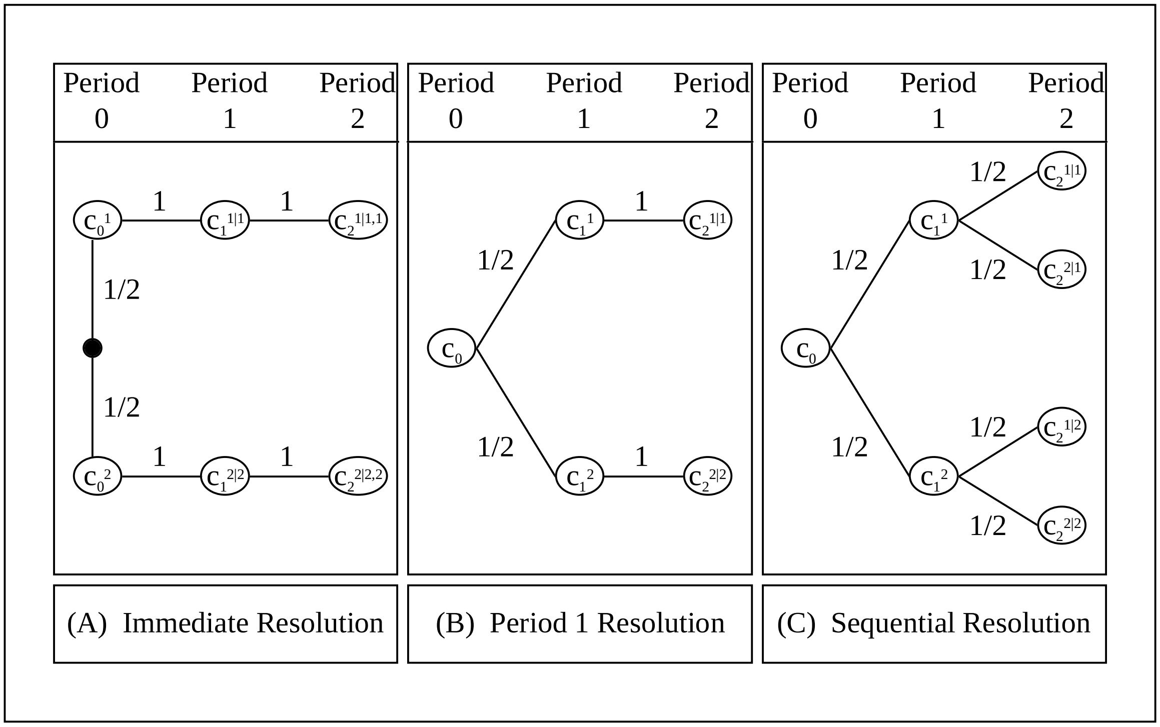

Figure 7-2. Consumption and resolution of uncertainty with expected utility.

The utility of these three formulations is determined by maximizing expected utility subject to the income constraints, as presented in Table 7.1. It is assumed that the decision maker can borrow and lend at the risk-free rate, where ψ_1^0 and ψ_2^0 correspond to the risk-free discount rate factors between period 0 and periods 1 and 2. Notice that that there is a change in some of the consumption variables as well as the constraints for the three formulations.

Table 7.2. Utility and constraints on formulations A, B, and C.

	Expected Utility	Constraints
A	$\frac{1}{2}\left[u_0\left(c_0^1\right)+u_1\left(c_1^{1\|1}\right)+u_2\left(c_2^{1\|1,1}\right)\right]$ $+\frac{1}{2}\left[u_0\left(c_0^2\right)+u_1\left(c_1^{2\|2}\right)+u_2\left(c_2^{2\|2,2}\right)\right]$	$c_0^1+\psi_1^0 c_1^{1\|1}+\psi_2^0 c_2^{1\|1,1} \le \psi_1^0 1.0+\psi_2^0 1.0$ $c_0^2+\psi_1^0 c_1^{2\|2}+\psi_2^0 c_2^{2\|2,2} \le \psi_1^0 0.3+\psi_2^0 0.3$
B	$u_0(c_0)+\frac{1}{2}\left[u_1\left(c_1^1\right)+u_2\left(c_2^{1\|1}\right)\right]$ $+\frac{1}{2}\left[u_1\left(c_1^2\right)+u_2\left(c_2^{2\|2}\right)\right]$	$c_0+\psi_1^0 c_1^1+\psi_2^0 c_2^{1\|1} \le \psi_1^0 1.0+\psi_2^0 1.0$ $c_0+\psi_1^0 c_1^2+\psi_2^0 c_2^{2\|2} \le \psi_1^0 0.3+\psi_2^0 0.3$
C	$u_0(c_0)+\frac{1}{2}\left[u_1\left(c_1^1\right)+\frac{1}{2}\left[u_2\left(c_2^{1\|1}\right)++u_2\left(c_2^{2\|1}\right)\right]\right]$ $+\frac{1}{2}\left[u_1\left(c_1^2\right)+\frac{1}{2}\left[u_2\left(c_2^{1\|2}\right)++u_2\left(c_2^{2\|2}\right)\right]\right]$	$c_0+\psi_1^0 c_1^1+\psi_2^0 c_2^{1\|1} \le \psi_1^0 1.0+\psi_2^0 1.0$ $c_0+\psi_1^0 c_1^1+\psi_2^0 c_2^{2\|1} \le \psi_1^0 1.0+\psi_2^0 0.3$ $c_0+\psi_1^0 c_1^2+\psi_2^0 c_2^{1\|2} \le \psi_1^0 0.3+\psi_2^0 1.0$ $c_0+\psi_1^0 c_1^2+\psi_2^0 c_2^{2\|2} \le \psi_1^0 0.3+\psi_2^0 0.3$

This example is made concrete by assuming that the decision maker's utility function is $U=\ln(z_0)+\ln(z_1)+\ln(z_2)$ and that the risk-free discount rate is 0 percent. Results from an optimization show that $U_A > U_C > U_B$.[17] That is, each of the three interpretations of the way uncertainty is resolved result in different utilities using an expected utility formulation. This confirms that an expected utility formulation can differentiate the way uncertainty is resolved into the analysis.

[17] The exact results are that U_A = -3.02, U_B = -3.26, U_C = -3.10.

8. Bibliography

Abel, Andrew B. 1983. "Optimal Investment Under Uncertainty." American *Economic Review* 73(1): 228-233.

Arrow, Kenneth J., and Anthony C. Fisher. 1974. "Environmental Preservation, Uncertainty, and Irreversibility." *The Quarterly Journal of Economics* 88 (May): 312-319.

Arrow, Kenneth J. 1968. "Optimal Capital Policy with Irreversible Investment." In *Value, Capital, and Growth. Papers in Honor Sir John Hicks*, edited by James N. Wolfe. Great Britain: Edinburgh University Press.

Bachelier, Louis. 1900. "Theory of Speculation." Translated from French in *The Random Character of Stock Market Prices*, revised edition, edited by Paul H. Coutner. Cambridge, Massachusetts: MIT Press.

Becker, Joao L., and Rakesh K. Sarin. 1989. "Decision Analysis Using Lottery-dependent Utility." *Journal Risk and Uncertainty*, 2(1): 105-117.

Bell, David E. 1995. "Risk, Return, and Utility." *Management Science* 41(1): 23-30.

Bertesekas, Dimitri. 1976. *Dynamic Programming and Stochastic Control*. New York: Academic Press.

Black, Fischer and Myron Scholes. 1973. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (May-June): 637-659.

Bodily, Samuel E. 1985. *Modern Decision Making: A Guide to Modeling with Decision Support Systems*, New York: McGraw-Hill Book Company.

Bonini, C. P. 1977 "Capital Investment Under Uncertainty with Abandonment Options." *Journal Financial and Quantitative Analysis* 12(1): 39-54.

Borison, A. 1996. "Portfolio Decision Analysis." Presentation at *INFORMS Conference*, November 3, Atlanta, GA.

Brealey, Richard A., and Stewart C. Myers. 1991. *Principles of Corporate Finance*, fourth edition. New York: McGraw-Hill.

Brennan, Michael J., and Eduardo S. Schwartz. 1985. "Evaluating Natural Resource Investments." *Journal of Business* 58(2): 135-157.

Caballero, Ricardo J. and Robert S. Pindyck. 1992. “Uncertainty, Investment, and Industry Revolution.” Working Paper No. 4160, National Bureau of Economic Research, Cambridge, Massachusetts. September.

Caballero, Ricardo J. 1991. “On the Sign of the Investment-Uncertainty Relationship.” *American Economic Review* 81(1): 279-288.

Chao, Hung-Po, and Robert Wilson. 1993. “Option Value of Emission Allowances.” *Journal of Regulatory Economics* 5: 233-249.

Chow, Gregory C. 1979. “Optimal Control of Stochastic Differential Equation Systems.” *Journal of Economic Dynamics and Control* 1: 143-175.

Chung, Kee H. 1990. “Output Decision Under Demand Uncertainty with Stochastic Production Function: A Contingent Claims Approach.” *Management Science* 36(11): 1311-1328.

Constantinides, George M.. 1978. “Market Risk Adjustment in Project Valuation.” *Journal Finance* 33(2): 603-616.

Copeland, T., T. Koller, and J. Murrin. 1994. *Valuation: Measuring and Managing the Value of Companies*, second edition. New York: John Wiley & Sons.

Cox, John C., Jonathan E. Ingersoll, Jr., and Stephen A. Ross. 1985. “An Intertemporal General Equilibrium Model of Asset Prices.” *Econometrica* 53(2): 363-384.

Cox, John C. and Stephen A. Ross. 1976. “The Valuation of Options for Alternative Stochastic Processes.” *Journal Financial Economics* 3(1/2): 145-166.

Cox, John C., Stephen A. Ross, and Mark Rubinstein. 1979. “Option Pricing: A Simplified Approach.” *Journal Financial Economics*, 7(3): 229-263.

Cox, John C. and Mark Rubinstein. 1985. *Options Markets*. Englewood Cliffs, New Jersey: Prentice Hall.

Coutner, Paul H. (editor). 1964. *The Random Character of Stock Market Prices*, revised edition. Cambridge, MA: MIT Press.

Debreu, Gerald. 1959. *Theory of Value*. New Haven: Yale University Press.

Dixit, Avinash K. and Robert S. Pindyck. 1994. *Investment Under Uncertainty*. New Jersey: Princeton University Press.

Dixit, Avinash. 1989. "Entry and Exit Decisions under Uncertainty." *Journal of Political Economy* 97(3): 620-638.

Dreze, J., and F. Modigliani. 1972. "Consumption Decisions Under Uncertainty." *Journal of Economic Theory* 5: 308-335.

Duffie, Darrell. 1992. *Dynamic Asset Pricing Theory*. Princeton, NJ: Princeton University Press.

Duffie, Darrell and Wayne Shafer. 1988. "Equilibrium and the Role of the Firm in Incomplete Markets / Darrell." Research Paper Number 915, Graduate School of Business, Stanford University.

Edwards, Ward. 1992. "Toward the Demise of Economic Man and Woman; Bottom Lines From Santa Cruz," in *Utility Theories: Measurements and Applications*, edited by Ward Edwards. Boston: Kluwer Academic Publishers.

Fishburn, Peter C. 1988. *Nonlinear Preference and Utility Theory*. Baltimore: Johns Hopkins University Press.

Fishburn, Peter C. 1982. *The Foundations of Expected Utility*. Boston: Kluwer Academic Press.

Geske, Robert. 1979. "The Valuation of Compound Options." *Journal of Financial Economics* 7: 63-81.

Gradshteyn, I. S., and I. M. Ryshik, edited by Alan Jeffrey. 1994. *Table of Integrals, Series, and Products*, fifth edition. San Diego: Academic Press.

Handa, Jagdish. 1977. "Risk, Probabilities, and a New Theory of Cardinal Utility." *Journal of Political Economy* 85(1): 97-122.

Harrison, Michael J. and David M. Kreps. 1979. "Martingales and Arbitrage in Multiperiod Securities Markets." *Journal Economic Theory* 20(3): 381-408.

Hassett, Kevin A., and Gilbert E. Metcalf. 1993. "Energy Conservation Investment: Do Consumers Discount the Future Correctly?" *Energy Policy* 21(6): 710-716.

Hassett, Kevin A., and Gilbert E. Metcalf. 1992. "Energy Tax Credits and Residential Conservation Investment." Working Paper No. 4020, National Bureau of Economic Research, Cambridge, Massachusetts. March.

Hershey, John C. and Paul. J. H. Schoemaker. 1985. "Probability versus Certainty Equivalence Methods in Utility Measurement: Are They Equivalent?" *Management Science* 31(10): 1213-1231.

Hodder, J. E. and H. E. Riggs. 1985. "Pitfalls in Evaluating Risky Projects." *Harvard Business Review* 63(1): 128-135.

Hogarth, Robin M. and Hillel J. Einhorn. 1990. "Venture Theory: A Model of Decision Weights." *Management Science* 36(7): 780-803.

Hoff, Thomas E. 1997a. "Using Distributed Resources to Manage Risks Caused by Demand Uncertainty." Submitted to *The Energy Journal*.

Hoff, Thomas E. 1997b. *Integrating Renewable Energy Technologies in the Electric Supply Industry: A Risk Management Approach*. National Renewable Energy Laboratory Report NREL/SR-520-23089, July.

Hoff, Thomas E. 1996. "Identifying Distributed Generation and Demand Side Management Investment Opportunities." *The Energy Journal* 17(4): 89-105.

Hoff, Thomas E. and Christy Herig. 1997. "Managing Risk Using Renewable Energy Technologies," *Symposium on the Virtual Utility, Topics in Regulatory Economics and Policy Series*. Kluwer Academic Publishers.

Hoff, Thomas. E., Howard J. Wenger, and Brian K. Farmer. 1996. "Distributed Generation: An Alternative to Electric Utility Investments in System Capacity." *Energy Policy* 24:2 137-147.

Howard, Ronald A. 1992. "In Praise of the Old Time Religion," in *Utility Theories: Measurements and Applications*, edited by Ward Edwards. Boston, MA: Kluwer Academic Publishers.

Howard, Ronald A. 1989. "Risk Preference." in *Readings on The Principles and Applications of Decision Analysis, Volume II*, edited by R. A. Howard and J. E. Matheson. Palo Alto: Strategic Decisions Group.

Howard, Ronald A. 1988. "Decision Analysis: Practice and Promise." *Management Science* 34(6): 679-695.

Huang, C. and Robert H. Litzenberger. 1988. *Foundations for Financial Economics*. Englewood Cliffs: Prentice Hall.

Hull, John C. 1993. *Options, Futures, and Other Derivative Securities*, second edition. Englewood Cliffs, New Jersey: Prentice Hall.

Ito, Kiyosi, and McKean, Henry P. Jr. 1974. *Diffusion Processes and their Sample Paths*, second printing. Berlin: Springer-Verlag.

Jarrow, Robert A. and Andrew Rudd. 1983. *Option Pricing*. Homewood: Dow Jones-Irwin.

Kahneman, Daniel and Amos Tversky. 1979. "Prospect Theory: An Analysis of Decision Under Risk." *Econometrica* 47(2): 263-291.

Karatzas, Ioannis, and Steven E. Shreve. 1991. *Brownian Motion and Stochastic Calculus*, second edition. New York: Springer-Verlag.

Karlin, Samuel, and Howard M. Taylor. 1975. *A First Course in Stochastic Processes*, second edition. New York: Academic Press.

Karlin, Samuel, and Howard M. Taylor. 1981. *A Second Course in Stochastic Processes*. New York: Academic Press.

Kasanen, E. and Lenos Trigeorgis. 1995. "Merging Finance Theory and Decision Analysis," in *Real Options in Capital Investment: Models, Strategies and Applications*, edited by Lenos Trigeorgis. Connecticut: Praeger.

Keeney, Ralph L., and Howard Raiffa. 1976. *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*. New York: John Wiley & Sons.

Keller, L. R. 1992. "Properties of Utility Theories and Related Empirical Phenomena," in *Utility Theories: Measurements and Applications*, edited by Ward Edwards. Boston: Kluwer Academic Publishers.

Kirkwood, Craig W. 1992. "An Overview of Methods for Applied Decision Analysis." *Interfaces* 22(6): 28-39.

Kreps, David M. 1988. *Notes on the Theory of Choice*. Boulder and London: Westview Press.

Kreps, David M., and Evan L. Porteus. 1978. "Temporal Resolution of Uncertainty and Dynamic Choice Theory." *Econometrica* 46(1): 185-200.

Kreps, David M., and Evan L. Porteus. 1979a. "Temporal von Neumann-Morgenstern and Induced Preferences." *Journal of Economic Theory* 20(1): 81-109.

Kreps, David M., and Evan L. Porteus. 1979b. "Dynamic Choice Theory and Dynamic Programming." *Econometrica* 47(1): 91-100.

Lam, Pok-sang. 1989. "Irreversibility and Consumer Durables Expenditures." *Journal of Monetary Economics* 23(1): 135-150.

LaValle, Irving H. and Peter C. Fishburn. 1996. "On the Varieties of Matrix Probabilities in Nonarchimedean Decision Theory." *Journal of Mathematical Economics* 25(1): 33-54.

LaValle, Irving H. 1992. "Small Worlds and Sure Things: Consequentialism by the Back Door," in *Utility Theories: Measurements and Applications*, edited by Ward Edwards. Boston: Kluwer Academic Publishers.

LaValle, Irving H. 1989. "New Choice Models Raise New Difficulties: Comment on Sarin," in *Organization and Decision Theory*, edited by I. Horowitz. The Netherlands: Kluwer-Nijhoff, Dordrecht.

LaValle, Irving H., and K. R. Wapman. 1986. "Rolling Back Decision Trees Requires the Independence Axiom." *Management Science* 32(3): 382-385.

Li, Shu. 1995. "Is There a Decision Weight π?" *Journal of Economic Behavior and Organization* 27: 453-463.

Luenberger, David G. 1997. *Investment Science*. Forthcoming from Oxford University Press.

Luenberger, David G. 1995. *Microeconomic Theory*. New York: McGraw-Hill, Inc.

Luenberger, David G. 1984. *Linear and Nonlinear Programming*, second edition. Reading, Massachusetts: Addison-Wesley Publishing Company.

Luenberger, David G. 1979. *Introduction to Dynamic Systems: Theory, Models, and Applications*. New York: John Wiley and Sons.

March, James G. 1978. "Bounded Rationality, Ambiguity, and the Engineering of Choice." *Bell Journal of Economics* 9:587-608.

Machina, Mark J. 1984. "Temporal Risk and the Nature of Induced Preferences." *Journal of Economic Theory* 33(2): 199-231.

Markowitz, Harry. 1952. "The Utility of Wealth." *Journal of Political Economy* :151-158.

Martzoukos, Spiros H., and Witold Teplitz-Sembitzky. 1992. "Optimal Timing of Transmission Line Investments in the Face of Uncertain Demand: An Option Valuation Approach." *Energy Economics* 14(1): 3-10.

Matheson, James E., and Ronald A. Howard. 1989. "An Introduction to Decision Analysis." in *Readings on The Principles and Applications of Decision Analysis, Volume I*, edited by R. A. Howard and J. E. Matheson. Palo Alto: Strategic Decisions Group.

McDonald, Robert, and Daniel Siegel. 1986. "The Value of Waiting to Invest." *The Quarterly Journal of Economics* 101(4): 707-727.

McDonald, Robert L., and Daniel R. Siegel. 1985. "Investment and the Valuation of Firms When There is an Option to Shut Down." *International Economic Review* 26(2): 331-349.

Merton, Robert C. 1971. "Optimal Consumption and Portfolio Rules in a Continuous-Time Model." *Journal of Economic Theory* 3(4): 373-413.

Merton, Robert C. 1973. "The Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science* 4: 141-183.

Miyamoto, John M. and Peter Wakker. 1996. "Multiattribute Utility Theory Without Expected Utility Foundations." *Operations Research* 44(2): 313-326.

Mossin, J. 1969. "A Note on Uncertainty and Preferences in a Temporal Context." *American Economic Review* 59(1): 172-174.

Olsen, Trond E., and Gunnar Stensland. 1991. "Invariant Controls in Stochastic Allocation Problems." In *Stochastic Models and Option Values*, edited by Diderik Lund and Bernt Oksendal. New York: North-Holland.

Olsen, Trond E., and Gunnar Stensland. 1989. "Optimal Sequencing of Resource Pools under Uncertainty." *Journal of Environmental Economics and Management* 17(1): 83-92.

Peck, Stephen C. 1974. "Alternative Investment Models for Firms in the Electric Utilities Industry." *The Bell Journal of Economics and Management Science* 5 (Autumn): 420-458.

Pettit, R. R. and R. F. Singer. 1986. "Instant Option Betas." *Financial Analysts Journal* 42(5): 51-62.

Pindyck, Robert S. 1993. "A Note on Competitive Investment Under Uncertainty." *American Economic Review* 83(1): 273-277.

Pindyck, Robert S. 1991. "Irreversibility, Uncertainty, and Investment." *Journal of Economic Literature* 29(3): 110-1152.

Pindyck, Robert S. 1988. "Irreversible Investment, Capacity Choice, and the Value of the Firm." *American Economic Review* 78(5): 969-985.

Pindyck, Robert S., and Daniel L. Rubinfeld. 1992. *Microeconomics*, second edition. New York: Macmillan Publishing Company.

Pope, R. 1985. "Timing Contradictions in von Neumann and Morgenstern's Axioms and Savage's 'Sure-Thing' Proof." *Theory and Decision* 18: 229-261.

Raiffa, Howard. 1968. *Decision Analysis: Introductory Lectures on Choices under Uncertainty*. Reading, MA: Addison-Wesley.

Robinson, Douglas R. 1991. *A Framework for Determining Optimal Petroleum Leasing*. Engineering-Economic Systems Department Dissertation, Stanford University, July.

Ross, Sheldon. 1988. *A First Course in Probability*, third edition. New York: Macmillan Publishing Company.

Ross, Stephen A. 1995. "Uses, Abuses, and Alternatives to the Net-Present-Value Rule." *Financial Management* 24(3): 96-102.

Rubinstein, Mark. 1992. "Double Trouble." *RISK*. January.

Rubinstein, Mark. 1991a. "Two into One." *RISK*. May.

Rubinstein, Mark. 1976. "The Valuation of Uncertain Income Streams and the Pricing of Options." *Bell Journal of Economics* 7(2): 407-425.

Rubinstein, Mark, and Eric Reiner. 1991b. "Breaking Down the Barriers." *RISK*. September.

Rust, John. 1987. "Optimal Replacement of GMC Bus Engines: An Empirical Model of Harold Zurcher." *Econometrica* 55: 999-1033.

Sarin, Rakesh, and Peter Wakker. 1994. "Folding Back in Decision Tree Analysis." *Management Science* 40(5): 625-628.

Smeloff, Ed, and Peter Asmus. 1997. *Reinventing Electric Utilities: Competition, Citizen Action, and Clean Power*, Washington D. C.: Island Press.

Smith, James E., and Robert F. Nau. 1995. "Valuing Risky Projects: Options Pricing Theory and Decision Analysis." *Management Science* 41(5): 795-816.

Smith, J. E. 1995. "Fisher Separation and Project Valuation in Partially Complete Markets." Fuqua School of Business working paper.

Spence, M., and R. Zeckhauser. 1972. "The Effect of the Timing of Consumption Decisions and the Resolution of Lotteries on the Choice of Lotteries." *Econometrica* 40(1): 401-403.

Sweeney, James L. 1993. "Economic Theory of Depletable Resources: An Introduction." In *Handbook of Natural Resource and Energy Economics*, edited by Allen V. Kneese and James L. Sweeney. Amsterdam: Elsevier Science Publishers.

Teisberg, Elizabeth O. 1995 "Methods for Evaluating Capital Investment Decisions Under Uncertainty," in Lenos Trigeorgis (ed.), *Real Options in Capital Investment: Models, Strategies and Applications*. Connecticut: Praeger.

Teisberg, Elizabeth O. 1993. "Capital Investment Strategies Under Uncertain Regulation." *RAND Journal of Economics* 24 (Winter): 591-604.

Teisberg, Elizabeth O. 1988. *Capital Investment Strategies Under Regulation: A Binomial Option Pricing Approach*. Engineering-Economic Systems Department Dissertation, Stanford University.

Triantis, Alexander J., and James E. Hodder. 1990. "Valuing Flexibility as a Complex Option." *The Journal of Finance* 45(5): 549-565.

Trigeorgis, Lenos. 1996. *Real Options: Managerial Flexibility and Strategy in Resource Allocation*. Cambridge, MA: MIT Press.

Trigeorgis, Lenos. 1995. *Real Options in Capital Investment*. Westport, Conn: Praeger.

Trigeorgis, Lenos and Scott P. Mason. 1987. "Valuing Managerial Flexibility." *Midland Corporate Finance Journal* 5(1): 14-21.

Trippi, Robert R., and Don. M. Chance. 1993. "Quick Valuation of the 'Bermuda' Capped Option." *The Journal of Portfolio Management* 20 (Fall): 93-99.

Tversky, Amos and Peter Wakker. 1995. "Risk Attitudes and Decision Weights." *Econometrica* 63(6): 1255-1280.

Tversky, Amos and Itamar Simonson. 1993. "Context-dependent Preferences." *Management Science* 39(10): 1179-1189.

Tversky, Amos and Daniel Kahneman. 1991. "Loss Aversion in Riskless Choice: A Reference-Dependent Model." *The Quarterly Journal of Economics* 106(4): 1039-1061.

Tversky, Amos and Daniel Kahneman. 1986. "Rational Choice and the Framing of Decisions." *Journal of Business* 59(4): S251-S277.

Varian, Hal R. 1992. *Microeconomic Analysis*, third edition. New York: W. W. Norton and Company.

Wakker, Peter. 1996. "The Sure-Thing Principle and the Comonotonic Sure-Thing Principle: An Axiomatic Analysis." *Journal of Mathematical Economics* 25:213-227.

Wasan, M. T. 1969. *First Passage Time Distribution of Brownian Motion with Positive Drift*. Department of Mathematics, Queen's University, Kingston, Ontario, Canada.

Wenger, Howard, Tom Hoff, and Jan Pepper. 1996. *Photovoltaic Economics and Markets: The Sacramento Municipal Utility District as a Case Study*. Report for California Energy Commission, Sacramento Municipal Utility District, and the U.S. Department of Energy.

Wilson, Robert B. 1968. "The Theory of Syndicates." *Econometrica* 40(1): 119-132.